Chiral spinors and gauge fields in noncommutative curved space-time

Nguyen Ai Viet *
Department of High-Energy Physics, CTP, Institute of Physics,
National Centre of Science and Technology, Hanoi, Vietnam
Kameshwar C. Wali †
Physics Department, Syracuse University
Syracuse, NY 13244-1130, U.S.A.

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Abstract

The fundamental concepts of Riemannian geometry, such as differential forms, vielbein, metric, connection, torsion and curvature, are generalized in the context of non-commutative geometry. This allows us to construct the Einstein-Hilbert-Cartan terms, in addition to the bosonic and fermionic ones in the Lagrangian of an action functional on non-commutative spaces. As an example, and also as a prelude to the Standard Model that includes gravitational interactions, we present a model of chiral spinor fields on a curved two-sheeted space-time with two distinct abelian gauge fields. In this model, the full spectrum of the generalized metric consists of pairs of tensor, vector and scalar fields. They are coupled to the chiral fermions and the gauge fields leading to possible parity violation effects triggered by gravity.

*E-mail:nguyenaiviet@iop.ncst.ac.vn
†E-mail:wali@physics.syr.edu

1 Introduction

It is widely recognized that our present concepts of space and time are inadequate to provide a satisfactory basis for a unified description of all elementary particle interactions including gravity. The twin pillars of modern physics, namely, 1). General Relativity, in which the dynamics of the classical continuum of space-time is coupled to the dynamics of the matter moving within it and 2). Quantum Field Theory, with rules of quantization to be applied, in principles, to all degrees of freedom including gravity, are found to be incompatible. While supersymmetric superstring theory has provided new insights, it is as yet far from being a convincing physical theory with predictable and experimentally verifiable consequences.

Both general relativity and quantum field theories assume that space-time is a continuum. A pseudo-Riemannian manifold, based on such a continuum picture, provides the basis for the geometrical description of the general theory of relativity. Likewise, quantum fields and their interactions are local operators that are functions of continuous, space-time coordinates. If one wants to explore the possibility of a unified quantum theory of space, time and matter, it appears that such a continuous space-time is inadequate. Recent ideas based both on string theory and approaches to quantum gravity suggest strongly that space-time structure at the Planck scale may be discrete with noncommutative coordinates. The continuum space-time of classical physics, it is hoped, will emerge in certain limiting regimes, just as the classical behavior of quantum systems emerges in an appropriate limit.

At the present time we do not have a precisely defined noncommutative space that meets the above requirements. However, in recent years, Connes has proposed an alternate approach to the study of the structure of space-time, based on noncommutative geometry (NCG) [1, 2][1]. It has given rise to the description of the Standard Model with a geometrical interpretation of the Higgs field on the same footing as the gauge field [6, 7]. Spontaneous symmetry breaking follows as a natural consequence. It enables one, in principle, to calculate some arbitrary parameters of the Standard Model, such as the Weinberg angle, the top quark and Higgs masses [8, 9, 10].

Connes' mathematical framework is general enough to explore also the character of noncommutative spaces at the energy scales that go beyond the

[1] see also [3, 4, 5] for a review

electro-weak scale. In this framework, Connes bypasses the precise specification of the manifold as the starting point. Instead, he formulates its description in terms of an associative and involutive algebra, commutative or noncommutative. One may think of this as a generalization of the well-known theorem due to Gelfand [1, 11], which states that the classical topological space based on a continuum can be completely recovered from the abelian algebra of smooth functions. We shall discuss Connes' ideas in some detail later in this paper, but we note here that his noncommutative geometry begins with what is called a spectral triple consisting of i) an involutive algebra $\mathcal{A}$ of operators, commutative or noncommutative, ii) a Hilbert space $\mathcal{H}$ as the carrier space for a representation of generalized, differential forms, constructed from this algebra, and iii) a self-adjoint operator D, called the Dirac operator, acting on $\mathcal{A}$. The algebra, $\mathcal{A}$, replaces and generalizes the commutative algebra of smooth functions. The Dirac operator D allows one to build a differential structure associated with any associative algebra and defines the metric.

Although the action functionals, in principle, can be constructed using the metric structure encoded in the Dirac operator, the procedure is still not unique. It allows different approaches to different problems. Thus, while the Diximier trace is used successfully to construct the action functional for the gauge sector of the Standard Model, other approaches have been pursued in the literature in dealing with the gravitational interactions. Thus, for instance, the Wodzicki residue is proposed to be used as an action functional for the gravity sector in the two-sheeted space-time. It leads to the conventional Einstein-Hilbert action supplemented by a term proportional to the square of a scalar field [12, 13]. The vector field that is part of the generalized metric is cancelled completely in the final action. The scalar field without a kinetic term does not propagate and causes problems when coupled to matter fields. The Diximier trace is used in other approaches with different generalizations of the Cartan formalism leading to a Brans-Dicke field coupled to gravity [14, 15]. In these approaches, the vector field vanishes as a consequence of a reality condition or a consistency condition. In a more recent paper [16], Connes and Chamseddine have proposed a new universal spectral action principle governing a noncommutative space that gives rise to the Einstein-Hilbert action together with higher order terms.

Landi, Viet and Wali [17] have adopted still another approach. It follows closely the standard Riemanian geometry and generalizes it in the case of a two-sheeted space-time that represents a discretized version of Kaluza-Klein theory in which, the fifth compact circular dimension is replaced by two discrete points. By formulating the basic notions of the conventional geometry in terms of algebraic forms, they define the latter in such a way that they make sense both in the case of commutative and noncommutative situations. In their simplified version of the model in which the generalized metric is assumed to be the same on the two sheets,the resulting action functional is exactly that of the zero mode sector of the Kaluza-Klein theory. In the more general case studied by Viet and Wali [18, 19], the Cartan formalism of pseudo-Riemannian geometry finds its generalization to allow the most general metric that contains pairs of tensor, vector and scalar fields. All of these fields are present in the resulting action functional with a set of constraints on the non-vanishing torsion that generalizes the torsion free condition in the standard space-time. They find a finite spectrum of the discretized Kaluza-Klein theory without requiring the truncation of the infinite mass spectrum. In the special case [19], when the torsion free condition is imposed, the spectrum reduces to the zero mode sector coupled to dilatons with a sixth order potential and a cosmological constant [19]. Recently, Viet [20] has found a new minimal set of constraints which together with the Cartan equations determine the connection, torsion and curvature in terms of the pairs of tensor, vector and scalar fields with a simpler Einstein-Hilbert-Cartan action functional.

In this paper, we extend our formalism to include a fermionic and its associated gauge boson sectors. As a prelude to the study of the Standard Model to include gravity, we consider a model of a left- and a right- chiral field living on a curved two-sheeted space-time and construct their action functionals. Due to the existence of different metric fields on the two sheets of space-time, we find a possible mechanism in our formalism for parity violation due to gravity.

In Section 2, we present the basic elements of the noncommutative differential geometry after a review of the algebraic approach to the standard differential geometry. This is primarily to set the stage for the contrasting NCG approach. The formulas of the key geometric notions in the standard case are formulated in such a way that they can be used as guidelines for generalizations in the noncommutative situation. In the Section 3., continuing the same approach, we introduce metric structure and construct Riemannian geometry using the algebraic approach and generalize it to the case of noncommutative geometry à la Connes. We also discuss in this section the representation mapping π given by Connes [1] to represent operators in a given Hilbert space. This procedure will be important in the realization of a noncommutative geometric model in the rest of the paper. In Section 4., we specialize to the algebra of differential forms and metric structure on the two-sheeted space-time. Here we follow the minimal constraints given in [20] to construct the Einstein-Hilbert-Cartan action. By following closely, the formulas given in Section 3, we establish a concrete realization of the abstract noncommutative geometry à la Connes in our formalism. In Section 5., we discuss the physical contents of the gravity sector in two-sheeted space-time. In Section 6., we construct the action for the gauge and chiral fermion sectors. Section 7. is devoted to discussions of some physical consequences of the model, and the final section is devoted to a summary and conclusions.

2 Algebraic approach to differential geometry

2.1 Global and algebraic approaches versus local construction

The traditional starting point of the standard differential geometry is the local construction of the vector spaces of tangent vectors and differential forms in the coordinate bases at an arbitrary local point $x \in \mathcal{M}$ [22, 23, 24]. In this construction, the differential forms are formulated as duals to the tangent vectors, which are derivatives of the functions at a point x. The differential forms form a vector space. All the geometric notions such as metric, connection, torsion, curvature and physical fields as scalars, vectors, tensors can be formulated conveniently in terms of differential forms. Although these notions can be defined locally, their meaning can only be understood clearly in the context of a global construction [24].

The physical fields that are defined at all points of a manifold $\mathcal{M}$, should be formulated strictly as the sections in a global framework of fiber bundles. In particular, the spinors are meaningful if and only if a global property of the manifold $\mathcal{M}$, the orientability, can be defined. The physical fields, which are formulated as sections of the fiber bundles with $\mathcal{M}$ as the base manifold, have an algebraic finite projective module structure. The connection together with other concepts such as the covariant derivative and the curvature can be formulated in the framework of this global construction [24]. This observation is important as it paves the way to a pure algebraic approach that allows direct generalization of the ordinary geometry to NCG without reference to the concept of a point.

The concept of a point as the starting point is the main obstacle to go beyond the standard geometry and seek a unified framework of general relativity and quantum theory. In an algebraic approach, on the other hand, it is possible to start from the pure algebraic structures to formulate the basic geometric elements. Thus, the Gelfand's theorem [11], which so far has been just an alternative formulation of differential geometry, becomes the central point in such an algebraic approach to geometry without the concept of a point.

A discussion of the global approach could provide a great deal of useful insight because it can serve as a bridge between the traditional and algebraic constructions. However, such a task is beyond the scope of this paper. [2] However, from the algebraic relations that are originally derived from a local construction of the ordinary geometry, the basic geometric and physical elements can be formulated in such a way that they can be generalized directly to NCG. In the following Sections, it is our goal to reformulate the basic notions of the ordinary geometry in the algebraic approach first, and then directly generalize them to the non-commutative case.

2.2 The ordinary differential geometry

The essential point of an algebraic approach is taking the algebra of function as a starting point. The set of differential forms will be considered as an algebraic module, a concept more general than that of a vector space, which also makes sense when the algebra is noncommutative.

The continuous functions $f(x) : \mathcal{M} \longrightarrow R$ form a commutative algebra $C^\infty(\mathcal{M})$.According to the Gelfand's theorem, all the information about the manifold $\mathcal{M}$ is encoded in this algebra. In other words, knowing the algebra $C^\infty(\mathcal{M})$ is equivalent to knowing the manifold $\mathcal{M}$ in the ordinary differential geometry.

[2] The interested readers are referred to [24] for more details in physical applications.

The graded algebra of differential forms is then constructed with the help of a differential operator d that maps the function f as a 0-form to the 1-form df. The module of 1-forms is extended by including all the elements $df.g$, where g is an arbitrary element of $C^\infty(\mathcal{M})$. The higher forms can be defined by extending the differential operator d as the mapping from n-forms onto $n+1$-forms.

The operator d is required to satisfy the following properties: i) $d^2 = 0$, ii) $d(ab) = da.b + (-1)^n a.db$, where a is a n-form and b is an arbitrary form (Leibnitz rule), iii) d is hermitian.

In a step analogous to the semi-classical limit of quantum physics, the algebra of functions, differential forms and the differential operator d can be treated as c-number operators acting on a Hilbert space $\mathcal{H}$ of wave functions.

In general, the differential operator can be given as any operator that satisfies the above three properties. In the particular case of ordinary differential geometry, the differential operator d is given in a finite basis of the 1-forms dx^μ $(\mu = 0, 1, ..., n)$,

$$d = dx^\mu \partial_\mu. \tag{2.1}$$

The assumption about the existence of a finite basis of a module of 1-forms is essential in the non-commutative case.

The action of d on a function f as a c-number operator is given as follows

$$df = [d, f] = dx^\mu \partial_\mu f(x). \tag{2.2}$$

An arbitrary 1-form u is also generated by the elements dx^μ

$$u = dx^\mu u_\mu, \tag{2.3}$$

where u_μ are contained in $C^\infty(\mathcal{M})$.

The action of d on u in accordance with the properties i)-iii) is

$$du = d(dx^\mu u_\mu) = d(dx^\mu).u_\mu - dx^\mu dx^\nu \partial_\nu u_\mu = -dx^\mu dx^\nu \partial_\nu u_\mu. \tag{2.4}$$

We can write du as a combination of an antisymmetric and a symmetric 2-forms,

$$du = \frac{1}{4}(dx^\mu dx^\nu - dx^\nu dx^\mu)(\partial_\mu u_\nu - \partial_\nu u_\mu) - \frac{1}{4}(dx^\mu dx^\nu + dx^\nu dx^\mu)(\partial_\mu u_\nu + \partial_\nu u_\mu). \tag{2.5}$$

The symmetric 1-forms must be dropped since they lead to the so called "junk forms". In a specific representation, a non-zero form can be derived from a form which is identical to zero. Junk forms are such forms. As an example, consider the form $df.f - f.df$ that is identical to zero; however its differential $d(df.f - f.df) = -df.df - df.df = -2df.df$ is not zero, if the symmetric form is allowed. The algebra of anti-symmetric 1-forms is a closed algebra by itself.

There are various ways to avoid junk forms. The simplest way, however, is to replace all an ordinary product by the wedge product

$$dx^\mu \wedge dx^\nu = \frac{1}{2}(dx^\mu dx^\nu - dx^\nu dx^\mu). \tag{2.6}$$

The Eqn.(2.6) can be generalized directly for the wedge product of an arbitrary number of the elements dx^μ just by anti-symmetrization.

A n-form ω is a purely antisymmetric tensor of the following form

$$\omega = dx^{\mu_1} \wedge ... \wedge dx^{\mu_n} \omega_{\mu_1 \dots \mu_n}. \tag{2.7}$$

The n-forms form a vector space $\Omega_x^n(\mathcal{M})$. The wedge product is defined between a m-form ω_1 and a n-form ω_2 to form a $m+n$-form $\omega_1 \wedge \omega_2$ as an anti-symmetrized tensor product

$$\wedge : \Omega_x^n(\mathcal{M}) \times \Omega_x^m(\mathcal{M}) \longrightarrow \Omega_x^{m+n}(\mathcal{M}). \tag{2.8}$$

One can define the graded algebra of the differential forms as $\Omega_x(\mathcal{M}) = \bigoplus_n \Omega_x^n(\mathcal{M})$.

As an example of differential forms, a Lie-algebra valued gauge field belonging to a group G or gauge connection b ,can be given as the 1-form

$$b = dx^\mu b_\mu, \tag{2.9}$$

with its field strength as the 2-form

$$f = db + b \wedge b = dx^\mu \wedge dx^\nu f_{\mu\nu} = dx^\mu \wedge dx^\nu \frac{1}{2}[(\partial_\mu b_\nu - \partial_\nu b_\mu) + [b_\mu, b_\nu]]. \tag{2.10}$$

As mentioned previously, in a "semi-classical" limit, all the forms are defined as operators on a Hilbert space $\mathcal{H}$ of spinor wave functions. The strict geometric formulation of the physical spinor fields is given as spinor bundles. On the Hilbert space $\mathcal{H}$ all the differential forms can be represented in a γ- matrix representation, if the abstract element dx^μ are represented as

$$dx^\mu = \gamma^\mu, \tag{2.11}$$

where γ^μ are the usual Dirac matrices.

In the γ-matrix representation, the differential operator d is represented by the Dirac operator

$$d = \gamma^\mu \partial_\mu. \tag{2.12}$$

The gauge field one-form and the field strength two-form are represented respectively by

$$b = \gamma^\mu b_\mu, \tag{2.13}$$

and

$$f = \sigma^{\mu\nu} f_{\mu\nu}, \tag{2.14}$$

where $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$.

Riemannian geometry can be constructed, in accordance with the Equivalence Principle, by introducing a locally flat orthonormal basis $\gamma^a, a = 0, 1, 2, 3$ and introducing a vierbein structure, which gives the transformation between the two frames

$$\begin{array}{rcl} \gamma^\mu & = & e^\mu_a \gamma^a \\ \gamma^a & = & e^a_\mu \gamma^\mu \\ e^a_\mu e^\mu_b & = & \delta^a_b \\ e^a_\mu e^\nu_a & = & \delta^\mu_\nu \end{array} \tag{2.15}$$

Gravity can be introduced via the connection 1-forms

$$\omega_{ab} = \gamma^c \omega_{abc}. \tag{2.16}$$

Two main notions that characterize the gravity structure in space-time, the torsion and curvature, can be defined by the Cartan's structure equations.

The torsion 1-forms t^a are given in the first structure equation

$$t^a = d\gamma^a - \gamma^b \wedge \omega^a_b \tag{2.17}$$

The curvature 2-forms $r^a{}_b$ are given in the second structure equation

$$r^a{}_b = d\omega_{ab} + \omega^a{}_c \wedge \omega^c{}_b, \tag{2.18}$$

which determines curvature completely in terms of connections.

2.3 Non-commutative differential geometry and the spectral triple

In the case of non-commutative geometry we shall follow the same steps as outlined in Sect 2.2. We will begin with Connes' spectral triple [1, 2], which consists of : 1) An involutive, unital algebra of functions $\mathcal{A}$, 2) Hilbert space $\mathcal{H}$, and 3) Dirac operator δ that satisfies the properties i)-iii)in Sect.2.2.

The existence of a finite basis for the module of differential forms is understood. Also, the algebra $\mathcal{A}$ of generalized functions is not necessarily commutative as in the conventional case. In this subsection, we shall begin with a brief review of Connes' abstract construction of universal enveloping algebra of the noncommutative differential forms.

The universal differential enveloping algebra constructed from a noncommutative algebra $\mathcal{A}$ is the differential algebra $\Omega^*(\mathcal{A})$, generated by all $a, b \in \mathcal{A}$ and a symbol δ, such that

$$\begin{aligned} \delta(1) &= 0, \\ \delta(ab) &= (\delta a)b + a(\delta b). \end{aligned} \tag{2.19}$$

By definition, the algebra of 0-forms $\Omega^0(\mathcal{A}) \equiv \mathcal{A}$; δa belongs to the space of universal 1-forms $\Omega^1(\mathcal{A})$, whose general element ω is a linear combination

$$\omega = \sum_i \delta a_i b_i, \qquad a_i, b_i \in \mathcal{A}. \tag{2.20}$$

The existence of a finite number of elements δa_i is guaranteed by a generalization of the vector space to the finite projective module. More details about this concept can be found in [1, 3]. However, here we just remark that in the ordinary geometry there is also a finite number of elements dx^μ which are the particular forms of δa_i.

In a vector space, scalar multiplication of a vector is defined with elements of an algebraic field, usually that of the complex or real numbers. The direct generalization of a vector space of 1-forms is an $\mathcal{A}$-module, where the linear combinations with real or complex coefficients are replaced by the elements of $\mathcal{A}$ and the scalar multiplication is defined with elements of the algebra $\mathcal{A}$ from right

$$\omega b = (\sum_i \delta a_i b_i) b = \sum_i \delta a_i (b_i b), \qquad \forall b \in \mathcal{A}. \tag{2.21}$$

The Leibnitzian property of δ, Eqn.(2.19), can be used to express left multiplication by an element of $\mathcal{A}$, $b\omega$,

$$\begin{array}{rcl} b\omega & = & b(\sum_i \delta a_i b_i) \\ & = & \sum_i \delta(ba_i)b_i - \delta b a_i b_i, \quad \forall a_i, b_i, b \in \mathcal{A}. \end{array} \tag{2.22}$$

That is to say, the generalized 1-forms form a $\mathcal{A}$-bimodule structure. Here, for the sake of definiteness, we have used the basis with δa_i on the left. However, it is always possible to use the Leibnitz rule (2.19) to rewrite a differential form with the right basis. Due to the non-commutativity of the algebra $\mathcal{A}$, the basis elements and the expansion coefficients in the right basis are not necessarily the same as in the left one as in the commutative case in Section 2.2.

To proceed further,we note that we can multiply two 1-forms as in the commutative case. The associativity of the algebra $\mathcal{A}$ and Eqn.(2.19) imply

$$(\delta b_0 a_0)(\delta b_1 a_1) = \delta b_0 (a_0 \delta b_1) a_1 = \delta b_0 \delta(a_0 b_1) a_1 - \delta b_0 \delta a_0 (b_1 a_1). \tag{2.23}$$

Continuing in this manner, we can write p-fold products of 1-forms, $\mathcal{A}$-coefficient linear combinations of which give the algebra $\Omega^p(\mathcal{A})$. A general element of $\Omega^p\mathcal{A}$ is of the form

$$\sum_i \delta a_{1i} ... \delta a_{pi} b_i, \quad a_{ki}, b_i \in \mathcal{A}. \tag{2.24}$$

The algebra $\Omega^p(\mathcal{A})$ is now a right $\mathcal{A}$-module being a generalization of the vector space of p-forms on a manifold. Clearly, by applying repeatedly Eqn.(2.22), one may define the product of a universal p-form with a q-form

$$\Omega^p\mathcal{A} \times \Omega^q\mathcal{A} \mapsto \Omega^{p+q}\mathcal{A}. \tag{2.25}$$

Hence, the space of universal forms has the structure of a graded algebra, the universal algebra $\Omega^*(\mathcal{A}) \equiv \bigoplus_p \Omega^p\mathcal{A}$. The involution on $\mathcal{A}$ extends uniquely to an involution on the algebra $\Omega^*\mathcal{A}$ when we impose the condition

$$(\delta a)^* \equiv -\delta a^* \tag{2.26}$$

To transform the graded algebra of forms $\Omega^*\mathcal{A}$ into a differential algebra, we consider the *differential* δ as a linear operator that takes

$$\Omega^p \longrightarrow \Omega^{p+1} \tag{2.27}$$

by defining

$$\delta(\delta a_1 ... \delta a_p b) \equiv (-1)^p \delta a_1 ... \delta a_p \delta b. \tag{2.28}$$

Eqn.(2.28) implies two basic relations,

$$\delta^2 \alpha = 0, \quad \forall \alpha \in \Omega^* \mathcal{A}, \tag{2.29}$$

$$\delta(\alpha_1 \alpha_2) = (\delta \alpha_1)\alpha_2 + (-1)^{deg\alpha_1} \alpha_1 \delta \alpha_2, \quad \forall \alpha_j \in \Omega^* \mathcal{A}. \tag{2.30}$$

Thus, all the constructions of the $\mathcal{A}$-module of differential forms are in exact parallel with the standard approach. In the next section, we shall present the general procedure to realize the above abstract construction in a given Hilbert space. We note that in the module of the generalized differential forms, the "junk" forms must be eliminated to guarantee that if $\delta\omega \neq 0$ the differential form ω must be a non-zero form.

2.4 Representation of the differential algebra on a given Hilbert space

The general constructions described in Sect 2.3 need to be realized by operators acting on appropriate Hilbert spaces in different applications. The operator representation of the differential forms is a direct generalization of the γ- matrix representation given in Sect. 2.2.

Connes [1, 2, 3] has given a general procedure to construct representations by the following graded homomorphism that preserves the involution,

$$\pi : \Omega^* \mathcal{A} \longrightarrow \mathcal{L}(\mathcal{H}),$$

$$\pi_p(\delta a_1 ... \delta a_p b) = \prod_{i=1}^{p} [D, \pi_0(a_i)] \pi_0(b), \tag{2.31}$$

where $\mathcal{L}(\mathcal{H})$ denotes the space of bounded operators on $\mathcal{H}$ and π_0, the representation of $\mathcal{A}$ on $\mathcal{H}$. The operator δ is represented by a self-adjoint operator D, the Dirac operator, on the Hilbert space $\mathcal{H}$, with compact resolvent, such that the commutator $[D, a]$ is a bounded operator, $\forall a \in \mathcal{A}$ [1].

In essence, differential forms, in the sense of Connes, are the images of universal forms under π. Strictly, however, a problem arises in the definition of these operator-valued forms, in that there exist universal forms α such that $\pi(\alpha) = 0$, but $\pi(\delta\alpha) \neq 0$. Once one mods out by these troublesome forms for each p, we obtain the space of operator-valued forms $\Omega^p_D(\mathcal{A})$, the analog of $\Omega^p(\mathcal{A})$. This is the problem of elimination of the "junk" forms in ordinary geometry. In two-sheeted space-time, as it will be shown in Sect. 4, this elimination can be done by a generalized wedge product in perfect parallelism with the conventional approach.

The various definitions proceed as in the previous subsection. We define

$$\Omega^*_D(\mathcal{A}) \equiv \bigoplus \Omega^p_D(\mathcal{A}) \tag{2.32}$$

Consider Eqn.(2.20)as a specific case to illustrate the representation given in Eqn.(2.31). The arbitrary 1-form ω can be represented as an operator U on the Hilbert space $\mathcal{H}$ as follows

$$U = \pi_1(\omega) = \pi_1(\delta a_i)\pi_0(b_i) = \Theta^i U_i, \tag{2.33}$$

where Θ^i are just the representations of the element δa_i in Eqn.(2.20). Thus,in this general context, the basis of differential forms is given without reference to a point. To be more specific with the Dirac operator D, let F be an operator representing a generalized function w in $\Omega^0_D(\mathcal{A}) \equiv \mathcal{A}$. From Eqn.(2.31) it follows that

$$\begin{aligned} DF &= [D, \pi_0(w)] = \pi_1(\delta w) = \pi_1(\delta a_i w_i) \\ &= \pi_1(\delta a_i)\pi_0(w_i) = \Theta^i (DF)_i. \end{aligned} \tag{2.34}$$

The coefficients $(DF)_i \in \mathcal{A}$ can now be regarded as derivatives but not referring to a specific point. Formally, we can define the operator D_i as

$$(DF)_i \equiv D_i F = [D_i, F]. \tag{2.35}$$

Hence, we can represent the Dirac operator in the Θ^i basis in the form

$$D = \Theta^M D_M. \tag{2.36}$$

From now on we use the latin upper case indices $M, N = 0, ..., n-1$ to replace the latin lower case indices i. Note that the expression of the Dirac operator D in terms of the operators D_M, implies a specific basis Θ^M. We are free to choose another basis of one-forms as a linear combination with coefficients $\in \mathcal{A}$. However, once the Dirac operator is given in a specific basis Θ^M, we can calculate derivation of forms in this basis and then transform the result to another basis whenever necessary.

Having outlined the construction of the operator algebra $\Omega_D^*(\mathcal{A})$, we shall henceforth abandon explicit mention of the distinction between universal forms and their representation as operators on $\mathcal{H}$, and denote them by capital letters.

3 Algebraic Formulation of Riemannian geometry within the framework of NCG

The construction of the universal algebra of differential forms based on a pseudo-Riemannian manifold follows the steps discussed in Sect.2. In its application to the theory of gravitation in general relativity, we need to introduce the concept of a metric, the affine connection, the inner products of one and two-forms, the torsion and curvature. In the NCG approach, the spectral triple consists of

The algebra $\mathcal{A}$: The algebra $C^\infty(\mathcal{M})$ of the analytic complex functions on the manifold $\mathcal{M}$.

The Hilbert space $\mathcal{H}$: Square integral functions that belong to the direct product of $\mathcal{A}$ and the spinor bundle $L^2(S, \mathcal{M})$.

Dirac operator D, whose representation in the Hilbert space $\mathcal{H}$ is to be specified.

The starting point of the general theory of relativity is the Equivalence Principle, which postulates that in an arbitrary gravitational field (curved space-time described by Riemannian geometry), it is possible to chose a locally inertial system (flat coordinate system), such that within a sufficiently small region, the laws of special relativity prevail. The transformation between the locally flat ortho-normal coordinate system and the general curved coordinate system introduces the well known vierbeins that can be regarded as gravitational fields.

The general Equivalence Principle allows the introduction of the metric via two sets of 1-forms Θ^A and Θ^M. The 1-forms Θ^A can be expanded in the Θ^M basis and vice versa,

$$\Theta^A = \Theta^M E_M^A, \quad \Theta^M = \Theta^A E_A^M. \tag{3.1}$$

E_M^A and E_A^M are elements of $\mathcal{A}$ satisfying

$$E_M^A E_B^M = \delta_B^A, \quad E_M^A E_A^N = \delta_N^M. \tag{3.2}$$

We define a metric tensor $\mathcal{G}^{MN}$ as the inner product of two 1-forms Θ^M and Θ^N. The inner product of 1-forms is defined as the sesqui-linear functional

$$<,>: \Omega_D^1 \times \Omega_D^1 \longrightarrow \mathcal{A},$$

and

$$\mathcal{G}^{MN} \equiv < \Theta^M, \Theta^N > . \tag{3.3}$$

$\mathcal{G}^{MN}$ are functions that are elements of the algebra $\mathcal{A}$, the algebra of infinitely smooth functions on the Riemannian manifold. As the algebra $\mathcal{A}$ is unital, η^{AB} (where $\eta^{AB} = diag(-1, 1, ..., 1)$) is also an element of $\mathcal{A}$. Therefore, we can choose the one-forms Θ^A so that they satisfy

$$\mathcal{G}^{AB} \equiv < \Theta^A, \Theta^B > = \eta^{AB}. \tag{3.4}$$

We reserve Latin uppercase A, B, ... to denote the "locally-flat" indices and M, N, ... to denote the "curved" or "derivative" indices in the basis Θ^M.

From Eqns.(3.3) and (3.4) we can derive the relation

$$\begin{aligned} < \Theta^A, \Theta^B > &= \eta^{AB} = < \Theta^M E_M^A, \Theta^N E_N^B > \\ &= \tilde{E}_M^A < \Theta^M, \Theta^N > E_N^B = \tilde{E}_M^A \mathcal{G}^{MN} E_N^B, \end{aligned} \tag{3.5}$$

where, $\tilde{E}_M^A$ denotes the involutive operation if it is defined for the algebra $\mathcal{A}$, otherwise it is E_M^A.

We note that the metric structure is determined if the curvilinear and locally orthonormal bases are given. The vierbein as a transformation matrix between the two bases determines the metric tensor. In the algebraic construction, it is more convenient to use these bases as a starting point to construct the Riemannian geometry.

If U is a general 1-form, $U = \Gamma^M U_M = \Gamma^A U_A$, the transformation between its components in the two coordinate systems is given by

$$U_M = E_M^A U_A \quad , \quad U_A = E_A^M U_M, \tag{3.6}$$

where E_A^M is the inverse matrix of E_M^A.

In particular, in the locally orthonormal basis, the exterior derivative of arbitrary forms must be first calculated in the curvilinear basis and then transformed back to the locally flat one by using the above transformations. For instance,

$$(DF)_A = E_A^M D_M F, \tag{3.7}$$

$$(DU)_{AB} = \frac{1}{2} E_A^M E_B^N (D_M U_N - D_N U_M). \tag{3.8}$$

The inner product of the basis 1-forms in Eqns.(3.3)and (3.5) generates the inner product in the vector space of 1-forms between $U = \Theta^M U_M$ and $V = \Theta^M V_M$ given by

$$< U, V >= \tilde{U}_M \mathcal{G}^{MN} V_N = \tilde{U}^N V_N. \tag{3.9}$$

The covariant derivative operator (affine connection), ∇, is given by the connection 1-forms Ω_B^A, where

$$\nabla \Theta^A = \Theta^B \bigotimes \Omega_B^A = \Theta^B \bigotimes \Theta^C \Omega_{BC}^A. \tag{3.10}$$

The Cartan structure equations define the torsion and curvature 2-forms of a given connection Ω_{AB} as follows:

$$T^A = D\Theta^A - \Theta^B \wedge \Omega_B^A, \tag{3.11}$$

$$R_B^A = D\Omega_B^A + \Omega_C^A \wedge \Omega_B^C. \tag{3.12}$$

In Eqns.(3.11) and (3.12), we need a definition of the wedge product of 1-forms. Such a definition, as explained in Sect.2, becomes specific when the representation mapping π is given. Here we only assume that the module of 2-forms is spanned by the formal wedge products $\Theta^A \wedge \Theta^B$ of 1-forms. For the ordinary space-time components, the anti-symmetrization is the standard convention. However, there is no guarantee that must be the case for more generalized components as shown in the case of the two-sheeted space-time in the Sect.4. In general, there are various ways to define the wedge product, depending on the specific applications.

The Cartan structure equations (3.11) and (3.12) are not sufficient to determine connection, torsion and curvature in terms of the vierbeins. In the conventional differential geometry, one generally imposes two additional constraints: i) the metric compatibility condition

$$\Omega_{AB} = -\Omega_{BA} \tag{3.13}$$

and ii) the torsion free condition

$$T^A = 0. \tag{3.14}$$

With these two constraints, the first structure equation determines the connection completely in terms of the vierbeins. The second structure equation, in turn, determines the curvature in terms of the connection.

To calculate the scalar curvature from the curvature two-form, Eqn.(3.12), the inner product of 2-forms is introduced as given by

$$<\Theta^A \wedge \Theta^B, \Theta^C \wedge \Theta^D> \equiv \eta^{AD}\eta^{BC} - \eta^{AC}\eta^{BD}. \tag{3.15}$$

This inner product can also be extended for two arbitrary 2-forms $G \equiv \Theta^A \wedge \Theta^B G_{AB}$ and $H \equiv \Theta^A \wedge \Theta^B H_{AB}$, $G_{AB}, H_{AB} \in \mathcal{A}$ as follows:

$$<G, H> = \tilde{G}_{AB}<\Theta^A \wedge \Theta^B, \Theta^C \wedge \Theta^D>H_{CD}. \tag{3.16}$$

Finally, we note that the Γ-matrix representation is given by the following homomorphism Γ

$$\Gamma(\Theta^A) = \Gamma^A, \qquad \Gamma(\Theta^M) = \Gamma^M, \tag{3.17}$$

where Γ^A are given as constant "flat" Dirac matrices. The concrete form of the "curved" Dirac matrices $\Gamma^M(x)$ is not known since it involves gravitational degrees of freedom through the vielbeins,

$$\Gamma^M(x) = E^M_A \Gamma^A. \tag{3.18}$$

Any specific assumption about the form of Γ^M independent of the above equation will be a restriction on the metric structure [3].

[3] In spite of this obvious fact, this kind of hidden assumption is often made in the literature without an explicit statement.

With the defining homomorphism (3.17) , we can transform the various formula in the Θ basis in the corresponding ones in the Γ-representation basis. Thus, for instance, the metric tensor is given by the inner product

$$G^{MN} = < \Gamma^M, \Gamma^N > = \frac{1}{4} Tr(\Gamma^M \Gamma^N), \tag{3.19}$$

where the trace is taken over the spinor indices. From Eqn.(3.5), the expression of the metric tensor in terms of the vierbeins is given as

$$\mathcal{G}^{MN} = \tilde{E}^M_A \eta^{AB} E^N_B. \tag{3.20}$$

Other details will be given in the next section where we use explicitly the Γ-matrix representation.

4 Geometry in two-sheeted space-time

In this section, we shall apply the general formalism developed in the previous section to the case of a two-sheeted space-time with a specific choice for the spectral triple.

4.1 Noncommutative differential forms in two-sheeted spacetime

Intuitively, two sheeted spacetime can be viewed as an extension of the physical four-dimensional spacetime manifold M by a discrete, internal space $\{a, b\}$. In other words, it is a discretized version of Kaluza-Klein theory [25] where the circle is replaced by two discrete points in the fifth dimension. This discretized Kaluza-Klein theory avoids the truncation inconsistency problem of the original theory since it has a finite field content with a finite mass spectrum [18, 19].

As discussed in the preceding sections, the requisite input for this model is the spectral triple, to which we apply the general techniques discussed in Sects 2.3 and 2.4. The algebra of smooth functions considered in Section 2.2 becomes $C^\infty(M, \mathbf{C})$ tensored with the complex-valued functions on the set $\{a, b\}$. Clearly any such function, $f : \{a, b\} \to \mathbf{C}$ can be written $f = f_a \oplus f_b$, each summand being isomorphic to $\mathbf{C}$ itself.

$$\mathcal{A} = C^\infty(\mathcal{M}) \otimes (\mathbf{C} \oplus \mathbf{C}) \cong C^\infty(\mathcal{M}, \mathbf{C}) \oplus C^\infty(\mathcal{M}, \mathbf{C}). \tag{4.1}$$

In our previous work, where we were concerned with only gravity [17, 19, 18], it was sufficient to have the algebra $C^\infty(R, \mathcal{M}) \bigoplus C^\infty(R, \mathcal{M})$. In the present paper, since we are also interested in matter fields coupled to gravity, we choose to begin with the more general algebra (4.1). However, we will find at the end, physics determines a subalgebra of (4.1) as physically relevant.

In Eqn.(4.1) we have essentially doubled the algebra presented in Sect.2.2, it is natural to take the representation space as the Hilbert space of square-integrable sections of a spinor bundle,

$$\mathcal{H} \equiv L^2(S, \mathcal{M}) \oplus L^2(S, \mathcal{M}). \tag{4.2}$$

The third element of the spectral triple is the generalized self-adjoint Dirac operator

$$D = \Gamma^M D_M, \quad M = 0, 1, 2, 3, 5 \tag{4.3}$$

with the generalization of the gamma matrices γ^μ given by

$$\Gamma^\mu \equiv \begin{pmatrix} \gamma^\mu & 0 \\ 0 & \gamma^\mu \end{pmatrix}, \qquad \Gamma^5 \equiv \begin{pmatrix} 0 & \gamma^5 \\ \gamma^5 & 0 \end{pmatrix}. \tag{4.4}$$

With the basis specified, it is now possible to give an explicit form of the generalized derivatives on the functions of the Hilbert space $\mathcal{H}$. The direct generalization of the ordinary spacetime derivative ∂_μ follows obviously,

$$D_\mu \equiv \begin{pmatrix} \partial_\mu & 0 \\ 0 & \partial_\mu \end{pmatrix}. \tag{4.5}$$

These derivatives act along each of the space-time sheets. The derivative between the two sheets is given as

$$D_5 \equiv \sigma \bar{D}_5, \tag{4.6}$$

where

$$\bar{D}_5 \equiv \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}, \qquad \sigma \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.7}$$

The parameter m has the dimension of mass in order to give the fifth component of spacetime the same dimension as the other components. If we formally write the operator DF as,

$$D(F) \equiv [D, F] = \Gamma^M (DF)_M = \Gamma^M D_M F, \tag{4.8}$$

we indeed obtain the derivative along the fifth dimension with the same dimension as other derivatives

$$D_5 F = \sigma[\bar{D}_5, F] = m(F - \tilde{F}), \tag{4.9}$$

where

$$\tilde{F} = \begin{pmatrix} f_2 & 0 \\ 0 & f_1 \end{pmatrix}. \tag{4.10}$$

As required for the spectral triple, the operator D on $\mathcal{H}$ is self-adjoint with respect to the inner product

$$<\psi|D\phi> = <D^\dagger\psi|\phi> \qquad , \qquad D = D^\dagger. \tag{4.11}$$

A straightforward definition of the generalized wedge product used in the previous work [17, 18, 19] is the fully anti-symmetric product that truncates all the "junk" forms in a trivial way, namely,

$$\Gamma^M \wedge \Gamma^N = \frac{1}{2}(\Gamma^M \Gamma^N - \Gamma^N \Gamma^M). \tag{4.12}$$

However, in this paper, since we would like to have the quartic Higgs potential in the gauge sector, we will adopt the wedge product with the following change in Eqn.(4.12), when $M, N = 5$,

$$\Gamma^5 \wedge \Gamma^5 = 1. \tag{4.13}$$

This extended definition of the wedge product can be used in noncommutative geometric constructions of the Standard Model. The emergence of a quartic potential, however, as we shall see later, requires the fifth dimension to be complex. This is more natural in the formalism of Viet [9, 10], where from the beginning the discretized fifth dimension is treated as complex. The complex Higgs fields emerge naturally and the fully anti-symmetric wedge product for all M, N can be retained.

With the given spectral triple, it is straightforward to construct the algebra of forms on the two-sheeted space-time. For later use, we work out the generalized γ-matrix representations of one-forms and their derivatives. The exterior derivative of any odd form is its anti-commutator with the operator D as a consequence of the Leibnitz rule for the generalized forms (2.30). Hence the exterior derivative DU of the generalized one-form $U = \Gamma^M U_M$ is

$$DU = \{D, U\} = \Gamma^M \wedge \Gamma^N (DU)_{MN}, \tag{4.14}$$

The components $(DU)_{MN}$ can be calculated with the help of the formulae (4.12) and (4.13) giving

$$\begin{array}{rcc} (DU)_{\mu\nu} & = & \frac{1}{2}(D_\mu U_\nu - D_\nu U_\mu) \\ (DU)_{\mu 5} & = & -(DU)_{5\mu} = \frac{1}{2}(D_\mu U_5 - D_5 U_\mu) \\ (DU)_{55} & = & m(U_5 + \tilde{U}_5). \end{array} \tag{4.15}$$

4.2 Riemannian geometry on a two-sheeted space-time

The Equivalence Principle extended to our two-sheeted space-time requires a locally orthonormal basis. Continuing to work with the Γ-representation, the locally orthonormal Θ^A given in Sect.3 are now represented by Γ^A,

$$\Gamma^a \equiv \begin{pmatrix} \gamma^a & 0 \\ 0 & \gamma^a \end{pmatrix}, \qquad \Gamma^{\dot{5}} \equiv \begin{pmatrix} 0 & \gamma^5 \\ \gamma^5 & 0 \end{pmatrix}, \tag{4.16}$$

where γ^a and γ^5 are the usual flat Dirac matrices. By choosing γ^5 we have specialized to the two sheets of spacetime of chiral spinors. We shall use a $\dot{5}$ index in the orthonormal basis to distinguish it from 5 in the general case.

The inner product now can be also taken as a trace over the Clifford indices. In the representation (4.16), the following orthogonality is manifest:

$$< \Gamma^A, \Gamma^B > \quad = \quad Tr(\Gamma^A \Gamma^B) \quad = \quad 2\eta^{AB}.\mathbf{1} \tag{4.17}$$

The curvilinear Θ^M will be represented by Γ^M. To obtain Γ^M and define the metric, we postulate the generalized vielbeins E^A_M, as the following diagonal matrix zero-forms:

$$E^\mu_a \equiv \begin{pmatrix} e^\mu_{1a} & 0 \\ 0 & e^\mu_{2a} \end{pmatrix}, \qquad E^\mu_{\dot{5}} \equiv 0 \ ,$$

$$E^5_a \equiv - \begin{pmatrix} a_{1a} & 0 \\ 0 & a_{2a} \end{pmatrix} \equiv -A_a = -E^\mu_a A_\mu \ ,$$

$$E_{\dot 5}^{5} \equiv \begin{pmatrix} \phi_1^{-1} & 0 \\ 0 & \phi_2^{-1} \end{pmatrix} \equiv \Phi^{-1} , \tag{4.18}$$

where $e_{1,2a}^{\mu}$ are two different vielbeins on the two sheets of space-time. Similarly, $a_{1,2}$ and $\phi_{1,2}$ are respectively vector and scalar fields. The dependence on the space-time variable ,x, of the above vielbeins is understood. In what follows, we shall, for the sake of simplicity of the formulas, continue to suppress the x-dependence.

The vielbeins E_A^M are invertible giving,

$$E_\mu^a \equiv \begin{pmatrix} e_{1\mu}^a & 0 \\ 0 & e_{2\mu}^a \end{pmatrix}, \qquad E_5^a \equiv 0 ,$$

$$E_\mu^{\dot 5} \equiv A_\mu \Phi , \qquad E_5^{\dot 5} \equiv \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix} \equiv \Phi . \tag{4.19}$$

Then, the Γ^M matrices are given as $\Gamma^M = \Gamma^A E_A^M$,

$$\Gamma^\mu \equiv \begin{pmatrix} \gamma^a e_{1a}^\mu & 0 \\ 0 & \gamma^a e_{2a}^\mu \end{pmatrix}, \qquad \Gamma^5 \equiv \begin{pmatrix} -a_1 & \gamma^5 \phi_2^{-1} \\ \gamma^5 \phi_1^{-1} & -a_2 \end{pmatrix}, \tag{4.20}$$

where $a_{1,2} = \gamma^a a_{1,2a}$ are two gauge connection one-forms on the two sheets.

Using the inner product defined as the trace on the Hilbert space, we can calculate the representation of the metric tensor as follows:

$$\begin{aligned} G^{MN} &= \tfrac{1}{2} Tr(\Gamma^M \Gamma^N) = \tfrac{1}{2} \tilde{E}_A^M Tr(\Gamma^A \Gamma^B) E_B^N \\ &= \tilde{E}_A^M \eta^{AB} E_B^N . \end{aligned} \tag{4.21}$$

In terms of the functions E_a^μ, A_μ and Φ the metric tensor can be expressed as,

$$\begin{aligned} G^{\mu\nu} &= G_0^{\mu\nu} \quad , \qquad G^{\mu 5} = -A^\mu, \\ G^{5\mu} &= -A^\mu \quad , \qquad G^{55} = A^2 + \Phi^{-2}, \end{aligned} \tag{4.22}$$

$$\begin{aligned} G_{\mu\nu} &= G_{0\mu\nu} + A_\mu A_\nu \Phi^2 \quad , \quad G_{\mu 5} = A_\mu \Phi^2, \\ G_{5\mu} &= A_\mu \Phi^2 \quad , \quad G_{55} = \Phi^2. \end{aligned} \tag{4.23}$$

The function G_0 is a generalized function with the metrics of the two space-time sheets on the diagonal

$$G_0^{\mu\nu} \equiv \begin{pmatrix} e_{1a}^\mu \eta^{ab} e_{1b}^\nu & 0 \\ 0 & e_{2a}^\mu \eta^{ab} e_{2b}^\nu \end{pmatrix}. \tag{4.24}$$

As we see from the trace of the generalized gamma matrices, no restrictions on the vielbeins can be derived if a consistent choice of basis is selected. We can only derive the formulae of generalized metric components in terms of the given vielbein components from Eqn.(4.21), which is in perfect analogy with the usual Riemannian geometry.

In terms of Γ-representations, an arbitrary generalized one-form U can be represented in both Γ-bases as follows:

$$U = \Gamma^A U_A = \Gamma^M U_M,$$

where U_M and U_A are diagonal matrix functions

$$U_A = \begin{pmatrix} u_{1A} & 0 \\ 0 & u_{2A} \end{pmatrix}, \qquad U_M = \begin{pmatrix} u_{1M} & 0 \\ 0 & u_{2M} \end{pmatrix}. \tag{4.25}$$

The functions U_M and U_A are related to each other by the transformation laws,

$$U_M = E^A_M U_A, \quad U_A = E^M_A U_M. \tag{4.26}$$

In terms of the functions E^a_μ, A_μ and Φ, the transformation laws can be written as

$$U_\mu = E^a_\mu U_a + A_\mu \Phi U_{\dot{5}}, \quad U_5 = \Phi U_{\dot{5}},$$

$$U_a = E^\mu_a (U_\mu - A_\mu U_5), \quad U_{\dot{5}} = \Phi^{-1} U_5. \tag{4.27}$$

The multiplication of two one-forms U and V is given by

$$UV = (\Gamma^A U_A).(\Gamma^B V_B) = (\Gamma^M U_M).(\Gamma^N U_N) \qquad (4.28).$$

Here we want to address two issues concerning the multiplication of 1-forms: Firstly, as we don't know the explicit form of the matrices Γ^M to rearrange the order of terms in NCG, it is convenient to carry out the multiplication in the Γ^A basis. In fact, using the explicit form of Γ^A in Eqn.(4.16), we can reorder the product in Eqn.(4.28). Secondly, as the explicit forms of the basis Γ^A are known, we can define the wedge product by anti-symmetrizing the product except for the component $\Gamma^{\dot{5}} \wedge \Gamma^{\dot{5}}$, which we will keep as a formal non-vanishing basis of the 'two-forms' algebra $\Omega^2(\mathcal{A})$. That is to say, we have to define the wedge product of the Γ^A matrices with the basis σ^{AB} of the 'two-forms' module $\Omega^2(\mathcal{A})$, to be given by

$$\sigma^{AB} = \Gamma^A \wedge \Gamma^B = \frac{1}{2}[\Gamma^A, \Gamma^B], \qquad (4.29).$$

when both A and B are not $\dot{5}$. And when $A = B = \dot{5}$ we have

$$\Gamma^{\dot{5}} \wedge \Gamma^{\dot{5}} \qquad (4.30).$$

as a formal basis vector. This component will generate a Higgs potential for the gauge fields.

From Eqn.(4.29), it is clear that

$$\Gamma^A \wedge \Gamma^B = -\Gamma^B \wedge \Gamma^A. \qquad (4.31).$$

except the product $\Gamma^{\dot{5}} \wedge \Gamma^{\dot{5}}$.

Now we know how to reorder the product in Eqn.(4.27) to define the wedge product of two 1-forms. The generalized functions commute with the matrices Γ^a. The commutation rule of a function F with the matrix $\Gamma^{\dot{5}}$ is as follows

$$F\Gamma^{\dot{5}} = \Gamma^{\dot{5}}\tilde{F}. \qquad (4.32).$$

It then follows that the product of two 1-forms U and V in Eqn.(4.28) can be defined as the wedge product

$$U \wedge V = \Gamma^A \wedge \Gamma^B (U \wedge V)_{AB}. \qquad (4.33).$$

and using the Eqns.(4.26), we can compute the component of the wedge product $U \wedge V$ in the locally flat frame,

$$\begin{array}{rcl} (U \wedge V)_{ab} & = & \frac{1}{2}(U_a V_b - U_b V_a) = \frac{1}{2}(E_a^\mu E_b^\nu - E_a^\nu E_b^\mu) \\ & & (U_\mu V_\nu + A_\mu (U_\nu V_5 - V_\nu U_5)), \\ (U \wedge V)_{\dot{5}a} & = & -(U \wedge V)_{a\dot{5}} = \frac{1}{2}(U_{\dot{5}} V_a - \tilde{U}_a V_{\dot{5}}) \\ & = & \frac{1}{2}\Phi^{-1}[U_5 E_a^\mu (V_\mu - A_\mu V_5) - \tilde{E}_a^\mu V_5 (\tilde{U}_\mu - \tilde{A}_\mu \tilde{U}_5)], \\ (U \wedge V)_{\dot{5}\dot{5}} & = & \tilde{U}_{\dot{5}} V_{\dot{5}} \quad = \quad (\Phi\tilde{\Phi})^{-1} \tilde{U}_5 V_5. \end{array} \qquad (4.34).$$

As a special case, we can calculate the wedge product of Γ^M one-forms as

$$\begin{array}{rcl} (\Gamma^M \wedge \Gamma^N)_{ab} & = & \frac{1}{2}(E_a^M E_b^N - E_a^N E_b^M), \\ (\Gamma^M \wedge \Gamma^N)_{\dot{5}a} & = & \frac{1}{2}(E_{\dot{5}}^M E_a^N - \tilde{E}_a^M E_{\dot{5}}^N), \\ (\Gamma^M \wedge \Gamma^N)_{\dot{5}\dot{5}} & = & \tilde{E}_{\dot{5}}^M E_{\dot{5}}^N. \end{array} \qquad (4.35).$$

The derivative DU of the 1-form U in the curved space-time basis follows from (4.15) and is given by

$$\begin{array}{rcl} DU & = & \Gamma^M \wedge \Gamma^N (DU)_{MN} = \Gamma^A \wedge \Gamma^B (DU)_{AB} \\ & = & \Gamma^A \wedge \Gamma^B (\Gamma^M \wedge \Gamma^N)_{AB} (DU)_{MN}, \end{array} \qquad (4.36).$$

and hence

$$\begin{array}{rcl} (DU)_{ab} & = & \frac{1}{2}(E_a^\mu E_b^\nu - E_a^\nu E_b^\mu)(D_\mu U_\nu + A_\mu (D_\nu U_5 - D_5 U_\nu), \\ (DU)_{\dot{5}a} & = & -(DU)_{a\dot{5}} = -\frac{1}{2}\Phi^{-1}[\frac{1}{2}(E_a^\mu + \tilde{E}_a^\mu)(D_\mu U_5 - D_5 U_\nu) \\ & & +m(E_a^\mu A_\mu - \tilde{E}_a^\mu \tilde{A}_\mu)(U_5 + \tilde{U}_5)], \\ (DU)_{\dot{5}\dot{5}} & = & -m(\Phi\tilde{\Phi})^{-1}(U_5 + \tilde{U}_5) \end{array} \qquad (4.37).$$

The inner product of two 2-forms

$$< \Gamma^A \wedge \Gamma^B, \Gamma^C \wedge \Gamma^D > \equiv \frac{1}{4} Tr_4(\Gamma^A \Gamma^B \Gamma^C \Gamma^D) = \eta^{AD}\eta^{BC} - \eta^{AC}\eta^{BD}. \quad (4.38).$$

Similarly, we can give a representation of the general p-forms, multiplication rules of a p-form and a q-form and the derivative of general p-forms. The use of the two frames is essential in such calculus. To summarize, it is convenient to carry out the multiplication in the Γ^A basis and the derivatives in the Γ^M basis and then transform to the Γ^A to obtain the desired components. The physical fields are given in the curved frame.

The definition of $\Gamma^{\dot{5}} \wedge \Gamma^{\dot{5}}$ is rather arbitrary but is required to retain the $\dot{5}\dot{5}$ component for a Higgs potential to survive in the gauge theory. It is more natural to introduce the discrete degrees of freedom as a pair of conjugate derivatives $D_z, D_{\bar{z}}$ as in [9, 10] than just a single D_5 in the Dirac operator.

In the gravitational sector, the curvature arises from a set of connection one-forms $\{\Omega_B^A \ A, B = 0, ...\dot{5}\}$. The connection coefficients are defined as those functions which arise in the expansion of Ω_B^A in the Γ-basis, as the generalization of (3.10),

$$\Omega_B^A \equiv \Gamma^C \Omega_{BC}^A. \qquad (4.39)$$

5 Physical content of the gravity sector in two-sheeted space-time

In Section 4., we have seen that the metric (vielbein) structure of the two-sheeted space-time contains a tensor, a vector and a scalar, which have generalized functions in Eqn.(4.18). This field content is exactly parallel to the one in the traditional Kaluza-Klein theory [25]. However, each generalized function has a pair of fields. Therefore, in principle, the gravity sector in the two-sheeted space-time, contains a pair of tensor, a pair of vector and a pair of scalar fields.

Intuitively as stated earlier, the two-sheeted space-time is a discretized version of the general untruncated Kaluza-Klein theory [26], where the internal space circle is replaced by two points. Therefore, intuition tells us that in general, we should expect one member of each pair to be massless and the other massive [18].

5.1 Minimal set of constraints and solutions of the first Cartan structure equation

The essential point is how to impose a consistent system of constraints to solve the generalized Cartan structure equations without overconstraining the metric. The second Cartan structure equation is used to calculate the curvature tensor in terms of the connections. Therefore, it does not have much to do with constraints. The first Cartan structure equation expresses a relationship between connections and torsion. Its component equations are as follows:

$$\begin{array}{rcl} T_{abc} &=& (D\Gamma_a)_{bc} + \frac{1}{2}(\Omega_{abc} - \Omega_{acb}) \\ T_{a\dot{5}b} &=& (D\Gamma_a)_{\dot{5}b} + \frac{1}{2}(\Omega_{a\dot{5}b} - \Omega_{ab\dot{5}}) \\ T_{\dot{5}ab} &=& (D\Gamma_{\dot{5}})_{ab} + \frac{1}{2}(\Omega_{\dot{5}ab} - \Omega_{\dot{5}ba}) \\ T_{\dot{5}\dot{5}b} &=& (D\Gamma_{\dot{5}})_{\dot{5}b} + \frac{1}{2}(\Omega_{\dot{5}\dot{5}b} - \Omega_{\dot{5}b\dot{5}}) \\ T_{ab\dot{5}} &=& (D\Gamma_a)_{b\dot{5}} + \frac{1}{2}(\Omega_{ab\dot{5}} - \Omega_{a\dot{5}b}) \\ T_{\dot{5}a\dot{5}} &=& (D\Gamma_{\dot{5}})_{a\dot{5}} + \frac{1}{2}(\Omega_{\dot{5}a\dot{5}} - \Omega_{\dot{5}\dot{5}a}) \\ T_{a\dot{5}\dot{5}} &=& (D\Gamma_a)_{\dot{5}\dot{5}} + \frac{1}{2}(\Omega_{a\dot{5}\dot{5}} + \tilde{\Omega}_{a\dot{5}\dot{5}}) \\ T_{\dot{5}\dot{5}\dot{5}} &=& (D\Gamma_{\dot{5}})_{\dot{5}\dot{5}} + \frac{1}{2}(\Omega_{\dot{5}\dot{5}\dot{5}} + \tilde{\Omega}_{\dot{5}\dot{5}\dot{5}}). \end{array} \tag{5.1}$$

Obviously these equations are not enough to determine both the torsion and the connection, even in the case of the ordinary geometry.

The torsion free condition $T^A = 0$ together with the metric compatibility condition turns out to be too restrictive giving only a theory without massive excitations. However, it is possible to impose constraints in such a way as to retain all the postulated fields in the metric structure in more than one way [18, 19, 20]. In this paper we will follow the results of [20] to impose the following minimal set of constraints:

i) Spacetime torsion free condition

$$T_{aBC} \quad = \quad 0, \tag{5.2.a}$$

ii) Metric compatibility condition

$$\Omega_{AB} \quad = \quad -\,\Omega_{BA} \tag{5.2.b}$$

iii) An additional condition on the connections.

$$\Omega_{AB\dot{5}} \quad = \quad 0 \tag{5.2.c}$$

With these conditions, Eqn.(5.1) can be solved giving the following solutions for the non-vanishing connection and torsion-tensor components,

$$\begin{array}{rcl}
\Omega_{abc} & = & -(D\Gamma_a)_{bc} + (D\Gamma_b)_{ac} - (D\Gamma_c)_{ba} \\
\Omega_{a\dot{5}b} & = & -2(D\Gamma_a)_{\dot{5}b} \\
\Omega_{\dot{5}ab} & = & 2(D\Gamma_a)_{\dot{5}b} \\
T_{\dot{5}bc} & = & (D\Gamma_{\dot{5}})_{bc} + (D\Gamma_b)_{\dot{5}c} - (D\Gamma_c)_{\dot{5}b} \\
T_{\dot{5}\dot{5}b} & = & (D\Gamma_{\dot{5}})_{\dot{5}b} \\
T_{\dot{5}\dot{5}\dot{5}} & = & (D\Gamma_{\dot{5}})_{\dot{5}\dot{5}}.
\end{array} \tag{5.3}$$

The righthand side of Eqns containing the components of $(D\Gamma_A)$ can be expressed in terms of metric components (vielbeins) and their derivatives by the rules already stated. Thus, we have non-vanishing torsion and connection coefficients in terms of metric fields.

Here let us note that no constraints are imposed on the metric components (vielbeins). Therefore, all the six fields will contribute to the scalar curvature. If all of them have appropriate kinetic terms in the final action, they will represent themselves as physical fields of the gravity sector of our theory.

However, the fields $e^{\mu}_{1,2\ a}$, $a_{1,2\ \mu}$ and $\phi_{1,2}$ turn out not to be good variables of the theory. They do not give rise to mass eigenstates except in the case $e_1 = e_2, a_1 = a_2$, when the theory reduces exactly to the zero mode sector of the Kaluza-Klein theory with massless fields.

In the next subsection, we define linear combinations of these fields which lead to appropriate fields leading to mass eigenstates.

5.2 New physical field variables

We begin with the following linear combinations as new physical field variables:

$$\begin{array}{rcl} e^{\mu}_{\pm a} & = & \frac{1}{2}(e^{\mu}_{1\ a} \pm e^{\mu}_{2\ \nu}), \\ a_{\pm\mu\nu} & = & \frac{1}{2}(a_{1\mu} \pm a_{2\mu}), \\ \phi_{\pm} & = & \frac{1}{2}(\phi_1 \pm \phi_2). \end{array} \tag{5.4}$$

When $e^{\mu}_{-\ \nu} = a_{-\ \mu} = \phi_{-} = 0$, the theory reduces to the usual Kaluza-Klein theory with $e^{\mu}_{+\ a}$ as the vierbein, $a_{+\mu}$ the vector field and ϕ_{+} the Brans-Dicke scalar.

To obtain the "physical" action functional in terms of the new variables, we need to define E^{μ}_{a} and its inverse E^{a}_{μ} in terms of the new variables. This is not as straightforward as one would think.

To this end, we start with

$$E^{\mu}_{a} = \begin{pmatrix} e^{\mu}_{1a} & 0 \\ 0 & e^{\mu}_{2a} \end{pmatrix} = e^{\mu}_{a}\mathbf{1} + v^{\mu}_{a}.\mathbf{r} \quad , \tag{5.5}.$$

where $v^{\mu}_{a} = e^{\mu}_{-a}$ and $e^{\mu}_{a} = e^{\mu}_{a+}$. Let $e^{a}_{\mu} = (e^{\mu}_{a})^{-1}$ and

$$\mathbf{1} \ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{r} \ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.6}.$$

From the orthogonality relation of the vielbeins

$$E^{\mu}_{a}E^{b}_{\mu} = \delta^{a}_{b}\mathbf{1} \tag{5.7}$$

we obtain

$$E^{a}_{\mu} = (e^{a}_{\mu} - v^{\nu}_{\mu}s^{a}_{\nu})\mathbf{1} + s^{a}_{\mu}\mathbf{r}, \tag{5.8}$$

where s^{a}_{μ} is a non-linear function of v^{a}_{μ} satisfying

$$v^{\nu}_{\mu} + s^{\nu}_{\sigma}(\delta^{\sigma}_{\mu} - v^{\lambda}_{\mu}v^{\sigma}_{\lambda}) = 0, \tag{5.9}$$

and $s^{\nu}_{\sigma} = e^{\nu}_{a}s^{a}_{\sigma}$.

The tensor field $v^{\mu\nu}$ is a candidate for a massive excitation of the massless symmetric metric tensor.

We can also use the metric tensors $g^{\mu\nu}$ and $g_{\mu\nu}$ to raise and lower indices and express the tensor fields v and s in a matrix notation, $v = [v^{\nu}_{\mu}] \quad , \quad s = [s^{\nu}_{\mu}]$.

The following algebraic relations are also valid

$$vs = sv \quad , \quad s = v(v^2 - 1)^{-1}. \tag{5.10}$$

The usual space-time component of the generalized vierbein E_a^μ and its inverse E_μ^a are determined completely in terms of the usual vierbeins e_a^μ, e_μ^a and the tensor field $v^{\mu\nu}$.

Furthermore, we can define the vector and scalar components of the metric by

$$\begin{array}{rcl} A_\mu & = & a_{+\ \mu}\mathbf{1} + a_{-\ \mu}\mathbf{r} \quad , \\ \Phi & = & \phi_+\mathbf{1} + \phi_-\mathbf{r} \quad . \end{array} \tag{5.11}$$

Hence, finally, the gravity sector of the theory contains e_a^μ, e_μ^a, the tensor field v_ν^μ, the vector and scalar fields $a_{\pm\mu}, \phi_\pm$. The tensor field s_ν^μ is not an independent field. It is a non-linear combination of v_μ^ν. As we will see in the final Lagrangian, the tensor field $v^{\mu\nu}$ will be a mass eigenstate. Hence we are led to make the assumption, $v^{\mu\nu} = v^{\nu\mu}$, in order to have $v^{\mu\nu}$ as an independent tensor field.

As we shall also see, the signature of the kinetic terms of the fields $v^{\mu\nu}$, a_μ and ϕ_- in the final Lagrangian built from curvature and torsion terms of our theory, requires the following field redefinitions,

$$\begin{array}{rcl} v_\mu^a & \longrightarrow & iv_\mu^a, \\ a_{-\mu} & \longrightarrow & ia_{-\mu}, \\ \phi_- & \longrightarrow & i\phi_-. \end{array} \tag{5.12}$$

This field redefinition will be used hereafter in this paper.

It is convenient to introduce the following quantities in further calculations:

$$\begin{array}{rcl} P_{a\mu\nu} & = & \partial_\mu(e_{a\nu} + s_{a\lambda}v_\nu^\lambda) - 2ma_{\nu-}s_{a\mu} \\ Q_{a\mu\nu} & = & \partial_\mu s_{a\nu} + 2ma_{\nu+}s_{a\mu} \\ X_{bc}^{[\mu,\nu]} & = & (e_b^\mu e_c^\nu - e_b^\nu e_c^\mu + v_b^\nu v_c^\mu - v_b^\mu v_c^\nu) \\ Y_{bc}^{[\mu,\nu]} & = & (e_b^\mu v_c^\nu - e_b^\nu v_c^\mu + v_b^\mu e_c^\nu - v_b^\nu e_c^\mu). \end{array} \tag{5.13}$$

In several formulae of this theory, $P_{a\mu\nu}$ and $Q_{a\mu\nu}$ generalize the field derivatives $\partial_\mu e_{a\nu}$ and $\partial_\mu v_{a\nu}$. The $X_{ab}^{[\mu,\nu]}$ generalize the usual term $e_a^\mu e_b^\nu - e_a^\nu e_b^\mu$ which is common in general relativity. Some useful contraction formulae for $X_{ab}^{[\mu,\nu]}$

and $Y_{ab}^{[\mu,\nu]}$ can be found in Appendix A.2. These formulae allow us to write the action in a compact form.

For calculational convenience, it is useful to introduce the following projection operations for the component $f_\pm$ of a generalized function F

$$\begin{array}{rcccl} (F)_+ &=& f_+ &=& \frac{1}{2}Tr(F) \quad , \\ (F)_- &=& f_- &=& \frac{1}{2}Tr(F.\mathbf{r}) \ . \end{array} \tag{5.14}$$

The following relations are also useful in practical calculations of products of generalized functions

$$\begin{array}{l} (F.G)_+ = (F)_+(G)_+ + (F)_-(G)_-, \\ (F.G)_- = (F)_+(G)_- + (F)_-(G)_+. \end{array} \tag{5.15}$$

5.3 Curvature, Torsion and Lagrangian

The second structure equation is generalized in the following form,

$$R_{AB} = D\Omega_{AB} + \eta^{CD}\Omega_{AC} \wedge \Omega_{DB}. \tag{5.16}$$

With the connections known from Eqn.(5.3), we can calculate all the components of the curvature in Eqn.(5.16). However, we are interested in only few components that contribute to the generalized scalar curvature

$$\begin{array}{rcl} R &=& \frac{1}{2}Tr < \Gamma^A \wedge \Gamma^B, R_{AB} >= R_{abcd+}\eta^{ad}\eta^{bc} + 2R_{a\dot{5}\dot{5}b+}\eta^{ab} \\ &=& (D\Omega_{ab})_{cd+}\eta^{ad}\eta^{bc} + (\Omega_{ae} \wedge \Omega_{bf})_{cd+}\eta^{ad}\eta^{cf}\eta^{be} \\ && +2(D\Omega_{a\dot{5}})_{\dot{5}b+}\eta^{ab} + 2(\Omega_{a\dot{5}} \wedge \Omega_{\dot{5}b})_{cd+}\eta^{ad}\eta^{bc}. \end{array} \tag{5.17}$$

The Lagrangian from curvature is defined as

$$\mathcal{L}_R = \frac{1}{16\pi G_N}R = \mathcal{L}_{R1} + \mathcal{L}_{R2} + \mathcal{L}_{R3}, \tag{5.18}$$

where

$$\begin{array}{rcl} \mathcal{L}_{R1} &=& (16\pi G_N)^{-1}(D\Omega_{ab})_{cd+}\eta^{ad}\eta^{bc} \\ \mathcal{L}_{R2} &=& (16\pi G_N)^{-1}(\Omega_{ae} \wedge \Omega_{bf})_{cd+}\eta^{ad}\eta^{cf}\eta^{be} \\ \mathcal{L}_{R3} &=& (8\pi G_N)^{-1}((D\Omega_{a\dot{5}})_{\dot{5}b+}\eta^{ab} + (\Omega_{a\dot{5}} \wedge \Omega_{\dot{5}b})_{cd+}\eta^{ad}\eta^{bc}), \end{array} \tag{5.19}$$

where G_N is the Newton gravitational constant. The detailed calculations of the above terms are given in Appendix A.3.

Examination of these results shows that the“curvature” Lagrangian contains the usual four-dimensional curvature (Einstein-Hilbert term) along with a full Lagrangian for the massive tensor field $v_{\mu\nu}$. It also contains mass and interaction terms for the scalar and vector components of the metric. However, it does not contain kinetic terms for the latter. In order to obtain the complete Lagrangian, we need to take into account the contribution from the non-vanishing components of the torsion. This is given by

$$\begin{array}{rcl} \mathcal{L}_T & = & -(16g_V^2)^{-1}Tr < T_A, T^A > \;=\; (8g_V^2)^{-1}Tr(\tilde{T}_{ABC}T^{ABC}) \\ & = & (4g_V^2)^{-1}(T_{ABC+}T_+^{ABC} - T_{ABC-}T_-^{ABC}) \\ & = & (4g_V^2)^{-1}(T_{\dot{5}bc+}T_+^{\dot{5}bc} - T_{\dot{5}bc-}T_-^{\dot{5}bc} + 2T_{\dot{5}\dot{5}b+}T_+^{\dot{5}\dot{5}b} - 2T_{\dot{5}\dot{5}b-}T_-^{\dot{5}\dot{5}b} + \; T_{\dot{5}\dot{5}\dot{5}+}T_+^{\dot{5}\dot{5}\dot{5}}) \\ & = & \mathcal{L}_{T1} + \mathcal{L}_{T2} + \mathcal{L}_{T3}, \end{array} \tag{5.20}$$

where the Lagrangians $\mathcal{L}_{T1}, \mathcal{L}_{T2}$ and $\mathcal{L}_{T3}$ are given in Appendix A.4. g_V is a constant with dimension of mass.

Finally, the Einstein-Hilbert-Cartan Lagrangian for the gravity sector in our theory is

$$\begin{array}{rcl} \mathcal{L}_{E-H-C} & = & \mathcal{L}_R + \mathcal{L}_T \quad = \quad (16\pi G)^{-1}TrR + (16g_V^2)^{-1}Tr < T_A, T^A > \\ & = & \mathcal{L}_{R1} + \mathcal{L}_{R2} + \mathcal{L}_{R3} + \mathcal{L}_{T1} + \mathcal{L}_{T2} + \mathcal{L}_{T3} \end{array} \tag{5.21}$$

The kinetic terms of the fields v_μ^a, $a_{\mu-}$ and ϕ_- in the resulting Lagrangian require the field redefinitions defined in Eqn.(5.12).

6 Matter fields in curved two-sheeted space-time

With the curved space-time described in the previous section as the background, we now proceed to construct Lagrangians for the gauge and fermionic sectors. The gauge sector will consist of two Abelian gauge fields with two Higgs scalar fields as part of a generalized one-form. The fermionic sector will consist of a left-chiral field on one sheet and a right-chiral field on the other.

6.1 The gauge sector

Let the one-form, U, be specialized to the gauge and Higgs fields to be associated with the chiral fermions,

$$U = B = \Gamma^M(x) B_M,$$

where

$$B_\mu = \begin{pmatrix} b_{1\mu} & 0 \\ 0 & b_{2\mu} \end{pmatrix} \quad , \quad B_5 = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \tag{6.1}$$

or

$$B_\mu = b_{\mu+}\mathbf{1} + b_{\mu-}\mathbf{r} \quad , \quad B_5 = h_+\mathbf{1} + h_-\mathbf{r},$$

where

$$b_{\pm\mu} = \frac{1}{2}(b_{1\mu} \pm b_{2\mu}) \quad , \quad h_\pm = \frac{1}{2}(h_1 \pm h_2). \tag{6.2}$$

Then, the curvature or field strengths are given by the two-form (henceforth the x-dependence will be understood),

$$G = DB + B \wedge B,$$

with components, following Eqns.(4.34)-(4.37),

$$G_{ab} = \frac{1}{2}(E^\mu_a E^\nu_b - E^\nu_a E^\mu_b)[D_\mu B_\nu + A_\mu(D_\nu B_5 - D_5 B_\mu)] \ , \tag{6.3}$$

$$\begin{aligned} G_{a\dot{5}} \quad = \quad & \tfrac{1}{2}E^5_{\dot{5}}[\tfrac{1}{2}(\tilde{E}^\mu_a + E^\mu_a)(D_\mu B_5 - D_5 B_\mu) + m(E^\mu_a A_\mu - \tilde{E}^\mu_a \tilde{A}_\mu)(B_5 + \tilde{B}_5) \\ & +(\tilde{E}^\mu_a \tilde{B}_\mu - E^\mu_a B_\mu + E^\mu_a A_\mu B_5 - \tilde{E}^\mu_a \tilde{A}_\mu \tilde{B}_5)B_5] \quad = \quad - G_{a\dot{5}}, \end{aligned} \tag{6.4}$$

$$G_{\dot{5}\dot{5}} = \tilde{E}^5_{\dot{5}} E^5_{\dot{5}}(m(\tilde{B}_5 + B_5) + \tilde{B}_5 B_5). \tag{6.5}$$

In terms of the above components, the Lagrangian is given by

$$\mathcal{L}_G = \frac{1}{2g^2} < G, G >= -\frac{1}{g^2}(\tilde{G}_{ab}G^{ab} + 2\tilde{G}_{a\dot{5}}G^{a\dot{5}} + \frac{1}{2}\tilde{G}_{\dot{5}\dot{5}}G^{\dot{5}\dot{5}}) \ . \tag{6.6}$$

As in the case of the torsion Lagrangian in the gravity sector, the gauge Lagrangian contains the kinetic terms for the fields $b_{-\mu}$ and h_- with a sign that requires a similar field redefinition. Therefore, we introduce the following new field variables in order to bring the gauge Lagrangian into a form, which is consistent with the standard model with two abelian gauge fields and a Higgs scalar.

$$\begin{array}{ccc} b_{\mu-} & \leftrightarrow & -ib_{\mu-}, \\ \eta & \leftrightarrow & h_+ + m + h_-, \\ \bar{\eta} & \leftrightarrow & h_+ + m - h_-, \\ b_{\mu\nu+} & = & \partial_\mu b_{\nu+} - \partial_\nu b_{\mu+}, \\ b_{\mu\nu-} & = & \partial_\mu b_{\nu-} - \partial_\nu b_{\mu-}, \\ \mathcal{D}_\mu \eta & = & \partial_\mu \eta - 2ib_{\mu-}\eta, \\ \mathcal{D}_\mu \bar{\eta} & = & \partial_\mu \bar{\eta} + 2ib_{\mu-}\bar{\eta}. \end{array} \tag{6.7}$$

Substituting the even and odd projection of G_{AB} from Eqns.(6.3)-(6.5) in Eqn.(6.6), we obtain the gauge sector Lagrangian $\mathcal{L}_G$. We write it in the form

$$\mathcal{L}_G = \mathcal{L}_{G1} + \mathcal{L}_{G2} + \mathcal{L}_{G3} + \mathcal{L}_{G4}, \tag{6.8}$$

where $\mathcal{L}_{G1}$ contains the kinetic terms of the vector gauge fields $b_{\mu+}$ and $b_{\mu-}$ and $\mathcal{L}_{G2}$ contains the kinetic terms of the Higgs field. $\mathcal{L}_{G3}$ contains the quartic Higgs fields. $\mathcal{L}_{G4}$ contains the remaining interaction terms of these fields with the scalar and vector fields of the gravity sector.

The Lagrangian (6.8), therefore, represent a generalization of the standard model gauge sector involving two abelian gauge fields coupled with a complex scalar with a quartic potential in the curved two-sheeted space-time. More discussion about the physical contents of this Lagrangian will be give in Sect.7.

6.2 Fermion sector

We begin with the conventional Dirac Lagrangian in curved space-time generalized to our two-sheeted space-time given by

$$\begin{array}{ccc} \mathcal{L}_F & = & i\bar{\Psi}\Gamma^A(E^M_A(D_M + iB_M) + \frac{i}{4}\Gamma^B\Gamma^C\Omega_{BCA})\Psi \\ & = & i\bar{\Psi}\Gamma^a[E^\mu_a(D_\mu + iB_\mu) + \frac{i}{4}\Gamma^b\Gamma^c\Omega^{(0)}_{bca} + \frac{i}{2}\Gamma^{\dot{5}}\Gamma^c\Omega_{\dot{5}ca}) \\ & & + \frac{i}{4}\Gamma^a\Gamma^b\Gamma^c\Omega^{(1)}_{bca} + (\Gamma^{\dot{5}}\Phi^{-1} - \Gamma^a E^\mu_a A_\mu)(D_5 + iB_5)]\Psi \\ & = & \mathcal{L}_{F1} + \mathcal{L}_{F2} + \mathcal{L}_{F3} + \mathcal{L}_{F4}, \end{array} \tag{6.9}$$

where

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix},$$

$$\psi_L = \frac{(1+\gamma_5)}{2}\psi \quad , \quad \psi_R = \frac{(1-\gamma_5)}{2}\psi,$$

ψ is the 4-component Dirac spinor and Ω_{BCA} are the connection coefficients. The expressions for the non-vanishing Ω_{ABC} (Ω_{abc}, $\Omega_{a\dot{5}b}$) are given in Eqn.(5.3). In terms of metric component fields and their derivatives, they are

$$\Omega_{abc} = \Omega^{(0)}_{abc} + \Omega^{(1)}_{abc},$$

where

$$\begin{aligned} \Omega^{(0)}_{abc} &= \tfrac{1}{2}(\eta_{ad}E^\mu_b E^\nu_c - \eta_{bd}E^\mu_a E^\nu_c + \eta_{cd}E^\mu_a E^\nu_b)(D_\mu E^d_\nu - D_\nu E^d_\mu), \\ \Omega^{(1)}_{abc} &= \tfrac{1}{2}(\eta_{ad}E^\mu_b E^\nu_c - \eta_{bd}E^\mu_a E^\nu_c + \eta_{cd}E^\mu_a E^\nu_b)(A_\nu D_5 E^d_\mu - A_\mu D_5 E^d_\nu), \end{aligned} \tag{6.10}$$

$$\Omega_{a\dot{5}b} = -\Omega_{\dot{5}ab} = 2\eta_{ad}E^5_{\dot{5}}E^\mu_b(D_5 E^d_\mu). \tag{6.11}$$

The separation of Ω_{abc} into $\Omega^{(0)}_{abc}$ and $\Omega^{(1)}_{abc}$ is motivated by the consideration of the fact that $\Omega^{(0)}$ contains only the metric vierbeins and their generalizations to the two-sheeted space-time, where as $\Omega^{(1)}$ contains the additional vector and scalar fields of the gravity sector.

The results of calculations of the Lagrangian $\mathcal{L}_F$ in Eqn.(6.9) are given in Appendix A.6.

7 Physical implications

So far, we have concentrated on developing the mathematical formalism within the framework of Conne's NCG. The various fields we have introduced do not have the desired physical dimensions. Just as we needed to redefine the fields (Eqns.(5.12) and (6.7)) to secure the coorect signs for the kinetic terms, we need to rescale them using the avialable dimensional parameters in our theory. To this end, we note that we have only three dimensional parameters, G_N the Newton gravitational constant, G_V the new gravitational constant from the torsion and the parameter m with dimensions of mass. The dimensionless gauge coupling g is the only other free parameter. By fixing the standard coefficients of the kinetic terms, we determine the various coupling and masses in terms of the four parameters G_N, G_V, m and g.

7.1 Kinetic terms and dimensions of the physical fields in the gravity sector

The Lagrangian of the gravity sector in Eqn.(5.21) contains the vierbeins e_a^μ that define the physical metric in the form

$$L_r = \frac{1}{16\pi G_N} r, \tag{7.1}$$

where r is the scalar curvature in the conventional Riemannian geometry.

We next consider the kinetic terms of the tensor field $v^{\mu\nu}$ ($s^{\mu\nu}$) by collecting together terms quadratic in these fields and quadratic in their derivatives. They lead to the partial Lagrangian

$$\begin{array}{rcl} L_v & = & -(4\pi G_N)^{-1}\nabla_\lambda v_{\mu\nu}\nabla^\lambda v^{\mu\nu} + (2\pi G_N)^{-1}\nabla_\lambda v_{\mu\nu}\nabla^\mu v^{\lambda\nu} \\ & & +(4\pi G_N)^{-1}\partial_\mu v_\nu^\nu \partial^\mu v_\rho^\rho - (2\pi G_N)^{-1}\partial_\mu v_\rho^\rho \nabla_\nu v^{\mu\nu} \\ & & -(4\phi G_N)^{-1} m^2 (v^{\mu\nu}v_{\mu\nu} - v_\mu^\mu v_\nu^\nu). \end{array} \tag{7.2}$$

To obtain the correct kinetic coefficient for $v^{\mu\nu}$, we need to define

$$v^{\mu\nu} \longrightarrow \sqrt{\pi G_N} v^{\mu\nu}. \tag{7.3}$$

With this redefinition, which incidently, gives the correct dimension to $v^{\mu\nu}$ (because of G_N) and yields the Fierz-Pauli Lagrangian for a spin-2 tensor field of mass m in curved spacetime.

The kinetic terms for the scalar fields are to be found from the Eqn.(A.4.6) in the Appdendix A.4. The field ϕ_+ needs special treatment as in the conventional Kaluza-Klein theory, since this field tends to 1 in vacuum. The following field redefinitions will give the fields correct physical dimensions,

$$\begin{array}{rcl} \phi_+ & \longrightarrow & \exp(2g_V\sigma) \\ \phi_- & \longrightarrow & 2g_V\phi_- \end{array} \tag{7.4}$$

The kinetic part of the scalar Lagrangian then becomes

$$\begin{aligned}\mathcal{L}_\phi \quad = \quad & -\tfrac{1}{2}(\exp(4g_V\sigma) + 4g_V^2\phi_-^2)^{-1}(\exp(2g_V\sigma)\partial_\mu\sigma\partial^\mu\sigma \\ & +\tfrac{1}{2}\partial_\mu\phi_-\partial^\mu\phi_-) - m^2(g_V)^{-1}(1+2g_V\phi_-^2\exp(-2g_V\sigma))^{-2},\end{aligned} \tag{7.5}$$

By expanding the non-linear factors in power of σ and ϕ_-^2 and retaining the lowest order terms, we obtain

$$\mathcal{L}_\phi \quad = \quad -\tfrac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \tfrac{1}{2}\partial_\mu\phi_-\partial^\mu\phi_- - m^2(g_V)^{-1}\phi_-^2 - m^2(g_V)^{-1}..., \tag{7.6}$$

which represents the kinetic part of the massless Brans-Dicke scalar field σ and the massive ϕ_-. It is to be noted that it also contains a cosmological constant term m^2/g_V.

Next, we consider the kinetic and mass terms for the vector fields $a_{\mu\pm}$ to be found in Eqn.(A.4.1)

$$\mathcal{L}_a = -\frac{1}{4g_V^2}(a_+^{\mu\nu}a_{\mu\nu+} + a_-^{\mu\nu}a_{\mu\nu-} + 2m^2 a_-^\mu a_{\mu-}) \tag{7.7}$$

With the field redefinition

$$a_{\mu\pm} \longrightarrow g_V a_{\mu\pm} \tag{7.8}$$

yields the massless vector field $a_{\mu+}$ and massive vector field $a_{\mu-}$ of mass $m/\sqrt{2}$.

The constant with dimension of mass g_V gives the scalar and vector fields correct physical dimensions in exactly the same way as the Newton gravitational G_N endows the correct dimension the massive tensor field $v^{\mu\nu}$.

7.2 Field redefinition in the gauge sector and gauge coupling

The vector and scalar fields in the gauge sector have the correct dimensions. However, those fields must also be redefined to have the standard kinetic terms:

$$\begin{aligned} b_{\mu\pm} \quad &\longrightarrow \quad g b_{\mu\pm} \\ \eta \quad &\longrightarrow \quad g\eta \\ \bar{\eta} \quad &\longrightarrow \quad g\bar{\eta} \end{aligned} \tag{7.9}$$

The partial gauge Lagrangian that contains only gauge vector and scalar fields comes from $\mathcal{L}_{G1}$, $\mathcal{L}_{G2}$, $\mathcal{L}_{G3}$ in the Appendix A.5. After redefinitions, it is given by

$$\begin{array}{rcl}\mathcal{L}_{b\eta} & = & -\frac{1}{4}b_{\mu\nu+}b_{+}^{\mu\nu} - \frac{1}{4}b_{\mu\nu-}b_{-}^{\mu\nu} \\ & & -(\phi_{+}^{2}+\phi_{-}^{2})^{-1}(\mathcal{D}^{\mu}\bar{\eta})(\mathcal{D}_{\mu}\eta) - \frac{g^{2}}{2}(\phi_{+}^{2}+\phi_{-}^{2})(\bar{\eta}\eta-(m/g)^{2})^{2}\end{array} \tag{7.10}$$

Implementing spontaneous symmetry breaking in the standard fashion, we find the mass of the surviving Higgs scalar to be $\sqrt{2}m$. The mass of $b_{\mu-}$ equals $2m$ and $b_{\mu+}$ remains massless.

From the fermionic Lagrangian (6.1) we find that the vector gauge fields $b_{\mu\pm}$ couple to the vector and axial vector currents $j_{\pm}^{\mu}$ and $j_{5\pm}^{\mu}$, where

$$\begin{array}{rcl} j_{+}^{\mu} & = & -g\bar{\psi}\gamma^{\mu}\psi \\ j_{5+}^{\mu} & = & -ig\sqrt{\pi G_{N}}\bar{\psi}\gamma^{\nu}\gamma^{5}v_{\nu}^{\mu}\psi \\ j_{-}^{\mu} & = & g\sqrt{\pi G_{N}}\bar{\psi}\gamma^{\nu}v_{\nu}^{\mu}\psi \\ j_{5-}^{\mu} & = & -ig\bar{\psi}\gamma^{\mu}\gamma^{5}\psi \end{array} \tag{7.11}$$

When the massive gravity field vanishes, as expected, the vector gauge field $b_{\mu+}$ couples only to the vector current j_{+}^{μ}, the vector gauge field $b_{\mu-}$ couples only to the axial current j_{5-}^{μ}.

The massive gravity field contributes a axial vector part to the current coupled to the $b_{\mu+}$ and a vector part to the current coupled to the $b_{\mu-}$. The magnitude of these contributions is determined by the Newton gravitational constant

7.3 Parity violating interactions due to extended gravity

The vector and axial vector currents of the fermionic fields coupled to the vector fields $a_{\mu\pm}$ of the gravity sector are given respectively by

$$\begin{array}{rcl} J_{+}^{\mu} & = & -m\pi G_{N}g_{V}\bar{\psi}\gamma^{a}Y_{ab}^{[\nu,\mu]}s_{\nu}^{b}\psi - \frac{1}{2}\sqrt{2}gg_{V}\bar{\psi}\gamma^{\mu}(\bar{\eta}+\eta)\psi \\ & & -\frac{i}{2}\sqrt{2\pi G_{N}}g_{V}\bar{\psi}\gamma^{\nu}v_{\nu}^{\mu}(\bar{\eta}-\eta)\psi \end{array} \tag{7.12}$$

and

$$\begin{array}{rcl} J_{5+}^{\mu} & = & im\pi G_{N}g_{V}\bar{\psi}\gamma^{a}\gamma^{5}X_{ab}^{[\nu,\mu]}s_{\nu}^{b}\psi - \frac{1}{2}\sqrt{2}gg_{V}\bar{\psi}\gamma^{\mu}\gamma^{5}(\bar{\eta}+\eta)\psi \\ & & +\frac{i}{2}\sqrt{2\pi G_{N}}g_{V}\bar{\psi}\gamma^{\nu}\gamma^{5}v_{\nu}^{\mu}(\bar{\eta}-\eta)\psi \end{array} \tag{7.13}$$

The vector current coupled to $a_{\mu-}$ is

$$J^{\mu}_{-} = -m\pi G_N g_V \bar{\psi}\gamma^a X^{[\nu,\mu]}_{ab} s^b_\nu \psi + \frac{i}{2}\sqrt{2} g g_V \bar{\psi}\gamma^\mu(\bar{\eta}-\eta)\psi - \frac{1}{2}\sqrt{2\pi G_N} g_V \bar{\psi}\gamma^\nu v^\mu_\nu(\bar{\eta}+\eta)\psi \tag{7.14}$$

The axial vector current coupled to $a_{\mu-}$ is

$$J^{\mu}_{5-} = -im\pi G_N g_V \bar{\psi}\gamma^a\gamma^5 Y^{[\nu,\mu]}_{ab} s^b_\nu \psi + \frac{i}{2}\sqrt{2} g g_V \bar{\psi}\gamma^\mu(\bar{\eta}-\eta)\psi + \frac{1}{2}\sqrt{2\pi G_N} g_V \bar{\psi}\gamma^\nu v^\mu_\nu(\bar{\eta}+\eta)\psi \tag{7.15}$$

8 Summary and conclusions

The main part of this paper is the geometric formulation of a two-sheeted space-time within the framework of Conne's NCG. To start with, we have reviewed algebraic formulation of the conventional differential geometry in order to set the stage for its generalization to noncommutative case.

Our starting point for the two-sheeted space-time is the choice of the algebra $\mathcal{A} = C^\infty(C, \mathcal{M}) \bigoplus C^\infty(C, \mathcal{M})$ as part of a spectral triple. However, our model leads us to a subalgebra where the two functions on two sheets are complex conjugates of each other. We are led to this subalgebra as the underlying mathematical structure of the two-sheeted space-time by physics. Our previous pure gravity theories on two-sheeted space-time [19, 18, 17] were constructed from a subalgebra of $\mathcal{A}$, which consisted of a pair of real functions. In the present paper, the construction of a spontaneously broken gauge theory requires that the fifth component of the gauge one-form must be a generalized complex valued function, with the values on the two sheets being complex conjugates of each other. To be mathematically consistent, the same condition need to be imposed on the differential forms of the theory. The algebra of the generalized function F of the form

$$F = \begin{pmatrix} f(x) & 0 \\ 0 & f^*(x) \end{pmatrix}, \tag{8.1}$$

where $f(x)$ is a complex valued function, forms a closed subalgebra of $\mathcal{A}$. Therefore, the noncommutative geometry can be constructed consistently with this subalgebra to describe the physics of chiral spinors and gauge fields coupled to gravity on two-sheeted space-time. Remarkably, from the stand-point of physics, the restriction to this subalgebra is also required by the signature of the kinetic terms of the massive modes in the theory. Thus, starting from a naturally general algebra, we have ended up with a restriction to a subalgebra to have a physically meaningful theory. Therefore, our results show that there is an intimate interplay between the mathematically consistent structure and physics in NCG.

The present theory also requires the definition of the wedge product to retain the Higgs quartic potential and to avoid the "junk forms" at the same time. The theory also requires a consistent involutive operator, that makes the theory consistent and the scalar products to be diagonal. The present theory also takes advantage of a minimal solution of the Cartan-Maurer structure equations, where the basic geometric quantities such as metric, connection, torsion and curvature are completely expressed in terms of the generalized vielbeins without imposing any constraint on its most general form. Physics requires that with such solutions, the contribution from torsion must be included to give kinetic terms to the physical fields. In other words, the mathematical structure of torsion is required by physics in this theory.

It is worth noting here that, although the above structure is rather unique in its simplicity and consistency, it is possible to construct the theory in an alternative way. For example, if the fully anti-symmetric wedge product is desired, one may resort to a Kahler structure of the internal space consisting of a pair of complex conjugates [9, 10]. In such a theory, the involutive operation can be chosen consistently to be the hermitian conjugate.

From the point of view of physics, the model presented here posseses a rich and complex strucvture that merits further exploration in many ways. The generalized gravity sector that includes massive tensor, vector and scalar fields, introduces simultaneous vector and axial vector couplings to matter fields. Consequently, we have the possibility of parity violating interactions due to gravity. In the context of a full standard model, we speculate that these interactions may provide the much sought CP-violating interactions in the early universe. This is further supported by the fact that such interactions involve G_N as the coupling. Consequently, we expect them to be small and hence correct order of magnitude.

In the context of cosmology, it is worth noting that the gravitational sector comes with a scalar field, an associated scalar potential and a cosmological constant term. A scalar field and a scalar potential are invoked in current literature to account for inflation and dark energy. Whether the potential in the model presented here has the desired features or not needs further exploration.

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APPENDIX

We collect together in this appendix some formulae used in our calculations in the main text. We also present various Lagrangians needed to construct the action functionals.The field redefinitions (5.12) and (6.7) are taken into account to obtain the correct signatures for the kinetic terms in the final actions.

A.1 Metric components $G^{\mu\nu}$ and $G_{\mu\nu}$

Expressed in terms of the physical variables defined in Sect.5.2, the even and odd components of the metric are given by

$$G^{\mu\nu} = G_{+}^{\mu\nu}\mathbf{1} + G_{-}^{\mu\nu}\mathbf{r} \quad , \quad G_{\mu\nu} = G_{\mu\nu+}\mathbf{1} + G_{\mu\nu-}\mathbf{r}, \tag{A.1.1}$$

where

$$G_{+}^{\mu\nu} = g^{\mu\nu} - g^{\sigma\lambda}v_{\sigma}^{\mu}v_{\lambda}^{\nu} \quad , \quad G_{-}^{\mu\nu} = i(g^{\mu\sigma}v_{\sigma}^{\nu} + g^{\nu\sigma}v_{\sigma}^{\mu}), \tag{A.1.2}$$

$$G_{\mu\nu+} = g_{\mu\nu} + (e_{\mu}^{a}\eta_{ab}s_{\lambda}^{b})v_{\nu}^{\lambda} + (e_{\nu}^{a}\eta_{ab}s_{\lambda}^{b})v_{\mu}^{\lambda} + (v_{\mu}^{\lambda}v_{\nu}^{\sigma} - \delta_{\mu}^{\lambda}\delta_{\nu}^{\sigma})s_{\lambda}^{a}\eta_{ab}s_{\sigma}^{b}, \tag{A.1.3}$$

$$G_{\mu\nu-} = i(e_{\mu}^{a}\eta_{ab}s_{\nu}^{b} + e_{\nu}^{a}\eta_{ab}s_{\mu}^{b} + v_{\mu}^{\lambda}s_{\lambda}^{a}\eta_{ab}s_{\nu}^{b} + v_{\nu}^{\lambda}s_{\lambda}^{a}\eta_{ab}s_{\mu}^{b}). \tag{A.1.4}$$

It is to be noted that $G^{\mu\nu}, G_{\mu\nu}$ are symmetric,

$$G_{\pm}^{\mu\nu} = G_{\pm}^{\nu\mu} \quad , \quad G_{\mu\nu\pm} = G_{\nu\mu\pm}. \tag{A.1.5}$$

A.2 Components of $D\Gamma^{A}$ From

$$\Gamma^{A} = \Gamma^{M}E_{M}^{A} \quad , \tag{A.2.1}$$

Eqn.(4.15) gives

$$\begin{array}{rcc} (D\Gamma^{A})_{\mu\nu} & = & \frac{1}{2}(D_{\mu}E_{\nu}^{A} - D_{\nu}E_{\mu}^{A}), \\ (D\Gamma^{A})_{\mu 5} & = & \frac{1}{2}(D_{\mu}E_{5}^{A} - D_{5}E_{\mu}^{A}) = -(D\Gamma^{A})_{5\mu} \\ (D\Gamma^{A})_{55} & = & m(E_{5}^{A} + \tilde{E}_{5}^{A}). \end{array} \tag{A.2.2}$$

Transforming Eqn.(A.2.2) into the locally flat frame, one obtains the components $(D\Gamma^{A})_{BC}$ as follows

$$
\begin{array}{rcl}
(D\Gamma^a)_{bc} &=& \frac{1}{2}E^\mu_b E^\nu_c[(D_\mu E^a_\nu - D_\nu E^a_\mu) + (A_\nu D_5 E^a_\mu - A_\mu D_5 E^a_\nu)], \\
(D\Gamma^a)_{b\dot{5}} &=& -\frac{1}{4}(D_5 E^a_\mu)E^5_{\dot{5}}(\tilde{E}^\mu_b + E^\mu_b) = -(D\Gamma^a)_{\dot{5}b}, \\
(D\Gamma^a)_{\dot{5}\dot{5}} &=& 0, \\
(D\Gamma^{\dot{5}})_{bc} &=& \frac{1}{2}E^\mu_b E^\nu_c[(D_\mu A_\nu - D_\nu A_\mu)\Phi + (A_\nu D_5(A_\mu \Phi) - A_\mu D_5(A_\nu \Phi))], \\
(D\Gamma^{\dot{5}})_{b\dot{5}} &=& \frac{1}{2}E^5_{\dot{5}}[\frac{1}{2}(\tilde{E}^\mu_b + E^\mu_b)(D_\mu E^{\dot{5}}_5 - D_5 E^{\dot{5}}_\mu) + m(\tilde{E}^5_b - E^5_b)(E^{\dot{5}}_5 + \tilde{E}^{\dot{5}}_5)], \\
(D\Gamma^{\dot{5}})_{\dot{5}\dot{5}} &=& m\tilde{E}^5_{\dot{5}}E^5_{\dot{5}}(E^{\dot{5}}_5 + \tilde{E}^{\dot{5}}_5).
\end{array}
\tag{A.2.3}
$$

We note that, since the lower indexed Γ_A matrix is defined as

$$\Gamma_A = \eta_{AD}\Gamma^D,$$

we use the following notation

$$(D\Gamma_A)_{BC} = \eta_{AD}(D\Gamma^D)_{BC}. \tag{A.2.4}$$

Using the definitions of the generalized vielbeins in Eqns.(4.18) and (4.19) and expressing them in terms of the new variables with the field redefinitions Eqn.(5.12) and (6.7), we find the even and odd components of $(D\Gamma^A)_{BC}$. They are

$$
\begin{array}{rcl}
(D\Gamma_a)_{bc+} &=& \frac{1}{2}[X^{[\mu,\nu]}_{bc}P_{a\mu\nu} - Y^{[\mu,\nu]}_{bc}Q_{a\mu\nu}] \\
(D\Gamma_a)_{bc-} &=& \frac{i}{2}[X^{[\mu,\nu]}_{bc}Q_{a\mu\nu} + Y^{[\mu,\nu]}_{bc}P_{a\mu\nu}],
\end{array}
\tag{A.2.5}
$$

$$
\begin{array}{rcccl}
(D\Gamma_a)_{b\dot{5}+} &=& -m\phi_- e^\mu_b s_{a\mu}(\phi_+^2 + \phi_-^2)^{-1} &=& -(D\Gamma_a)_{\dot{5}b+}, \\
(D\Gamma_a)_{b\dot{5}-} &=& -im\phi_+ e^\mu_b s_{a\mu}(\phi_+^2 + \phi_-^2)^{-1} &=& -(D\Gamma_a)_{\dot{5}b-},
\end{array}
\tag{A.2.6}
$$

$$(D\Gamma^a)_{\dot{5}\dot{5}\pm} = 0 \tag{A.2.7}$$

$$
\begin{array}{rcl}
(D\Gamma_{\dot{5}})_{bc+} &=& \frac{1}{2}(X^{[\mu,\nu]}_{bc}(a_{\mu\nu+}\phi_+ - a_{\mu\nu-}\phi_-) - Y^{[\mu,\nu]}_{bc}(a_{\mu\nu+}\phi_- + a_{\mu\nu-}\phi_+)) \\
&& -m(X^{[\mu,\nu]}_{bc}a_{\nu-}a_{\mu+}\phi_- + Y^{[\mu,\nu]}_{bc}a_{\nu+}a_{\mu-}\phi_+) \\
(D\Gamma_{\dot{5}})_{bc-} &=& \frac{i}{2}[(X^{[\mu,\nu]}_{bc}(a_{\mu\nu+}\phi_- + a_{\mu\nu-}\phi_+) + Y^{[\mu,\nu]}_{bc}(a_{\mu\nu+}\phi_+ - a_{\mu\nu-}\phi_-)) \\
&& +2m(X^{[\mu,\nu]}_{bc}a_{\nu+}a_{\mu-}\phi_+ - Y^{[\mu,\nu]}_{bc}a_{\nu-}a_{\mu+}\phi_-)],
\end{array}
\tag{A.2.8}
$$

$$
\begin{array}{rcl}
(D\Gamma_{\dot{5}})_{b\dot{5}+} &=& (\phi_+^2 + \phi_-^2)^{-1}[\frac{1}{2}(\phi_+\partial_\mu\phi_+ + \phi_-\partial_\mu\phi_-)e^\mu_b \\
&& -me^\mu_b\phi_-(a_{\mu-}\phi_+ + a_{\mu+}\phi_-) + 2m\phi_-\phi_+(a_{\mu-}e^\mu_b + a_{\mu+}v^\mu_b)] \\
(D\Gamma^{\dot{5}})_{b\dot{5}-} &=& i(\phi_+^2 + \phi_-^2)^{-1}[\frac{1}{2}(\phi_+\partial_\mu\phi_- - \phi_-\partial_\mu\phi_+)e^\mu_b \\
&& -me^\mu_b\phi_+(a_{\mu-}\phi_+ + a_{\mu+}\phi_-) + 2m\phi_+^2(a_{\mu-}e^\mu_b + a_{\mu+}v^\mu_b)],
\end{array}
\tag{A.2.9}
$$

$$(D\Gamma_{\dot{5}})_{\dot{5}\dot{5}+} \;=\; 2m\phi_+(\phi_+^2+\phi_-^2)^{-1} \quad , \quad (D\Gamma^{\dot{5}})_{\dot{5}\dot{5}-} = 0, \qquad (A.2.10)$$

where we have defined

$$a_{\mu\nu\pm} = \frac{1}{2}(\partial_\mu a_{\nu\pm} - \partial_\nu a_{\mu\pm}), \qquad (A.2.12)$$

and $P_{a\mu\nu}$, $Q_{a\mu\nu}$, $X_{ab}^{[\mu,\nu]}$, $Y_{ab}^{[\mu,\nu]}$ are defined in Eqn.(5.13).

The following formulae for $X_{bc}^{[\mu,\nu]}$ and $Y_b^{[\mu,\nu]}c$ will be useful in the torsion and gauge Lagrangian calculations

$$\begin{aligned} X_{bc}^{[\mu,\nu]}X^{bc[\rho,\tau]} &= 2(g^{\mu\rho}g^{\nu\tau} - g^{\mu\tau}g^{\nu\rho} + 2v^{\mu\tau}v^{\nu\rho} - 2v^{\mu\rho}v^{\nu\tau} \\ &+ < v^\mu, v^\rho >< v^\nu, v^\tau > - < v^\mu, v^\tau >< v^\nu, v^\rho >) \end{aligned} \qquad (A.2.13)$$

$$\begin{aligned} X_{bc}^{[\mu,\nu]}Y^{bc[\rho,\tau]} &= 2(g^{\mu\rho}v^{\nu\tau} - g^{\mu\tau}v^{\nu\rho} + v^{\mu\rho}g^{\nu\tau} - v^{\mu\tau}g^{\nu\rho} + v^{\nu\rho} < v^\mu, v^\tau > \\ &\quad -v^{\nu\tau} < v^\mu, v^\rho > +v^{\mu\tau} < v^\nu, v^\rho > -v^{\mu\rho} < v^\nu, v^\tau >), \end{aligned} \qquad (A.2.14)$$

$$\begin{aligned} Y_{bc}^{[\mu,\nu]}Y^{bc[\rho,\tau]} &= 2(g^{\mu\rho} < v^\nu, v^\tau > -g^{\mu\tau} < v^\nu, v^\rho > +g^{\nu\tau} < v^\mu, v^\rho > \\ &\quad -g^{\nu\rho} < v^\mu, v^\tau > +2v^{\mu\rho}v^{\nu\tau} - 2v^{\mu\tau}v^{\nu\rho}) \end{aligned} \qquad (A.2.15)$$

Using the formulae (A.2.1)-(A.2.1), it is straightforward to calculate all components of the connection, torsion and scalar curvature.

A.3 Curvature Lagrangian

For the calculation of the scalar curvature in Eqn.(5.17) the components Ω_{abc} and $\Omega_{a\dot{5}b}$ of the connection Ω_{AB} are necessary. The explicit expressions of the projections of these components are given by

$$\begin{aligned} \Omega_{a\dot{5}b+} &= -2(D\Gamma_a)_{\dot{5}b+} = -2m\phi_- e_b^\mu s_{a\mu}(\phi_+^2+\phi_-^2)^{-1} \\ \Omega_{a\dot{5}b-} &= \;\; 2(D\Gamma_a)_{\dot{5}b-} = 2im\phi_+ e_b^\mu s_{a\mu}(\phi_+^2+\phi_-^2)^{-1} \end{aligned} \qquad (A.3.1)$$

$$\begin{aligned} \Omega_{abc+} &= -((D\Gamma_a)_{bc+} - (a\leftrightarrow b) + (a\leftrightarrow c)) \\ &= \quad -\tfrac{1}{2}((X_{bc}^{[\mu,\nu]}P_{a\mu\nu} - Y_{bc}^{[\mu,\nu]}Q_{a\mu\nu}) - (a\leftrightarrow b) + (a\leftrightarrow c)) \\ \Omega_{abc-} &= -((D\Gamma_a)_{bc-} - (a\leftrightarrow b) + (a\leftrightarrow c)) \\ &= \quad -\tfrac{i}{2}((X_{bc}^{[\mu,\nu]}Q_{a\mu\nu} + Y_{bc}^{[\mu,\nu]}P_{a\mu\nu}) - (a\leftrightarrow b) + (a\leftrightarrow c)), \end{aligned} \qquad (A.3.2)$$

where $(a \leftrightarrow b)$ and $(a \leftrightarrow b)$ are permutations of the indices of the given tensors.

Now we can calculate the following terms

$$\begin{array}{rcl}(D\Omega_{ab})_{cd+}\eta^{ad}\eta^{bc} & = & -X^{ab[\mu,\nu]}(\frac{1}{2}\partial_\mu\Omega_{ab\nu+} + ima_{\nu-}\Omega_{ab\mu-}) \\ & & -iY^{ab[\mu,\nu]}(\frac{1}{2}\partial_\mu\Omega_{ab\nu-} + ima_{\nu-}\Omega_{ab\mu+}),\end{array} \tag{A.3.3}$$

where $\Omega_{ab\mu\pm}$ can be expressed in terms of $\Omega_{abc\pm}$ as follows

$$\begin{array}{rcl}\Omega_{ab\mu+} & = & (e^c_\mu + e^c_\nu s^{\nu\rho}\upsilon_{\rho\mu})\Omega_{abc+} + is^c_\mu\Omega_{abc-} \\ \Omega_{ab\mu-} & = & (e^c_\mu + e^c_\nu s^{\nu\rho}\upsilon_{\rho\mu})\Omega_{abc-} + is^c_\mu\Omega_{abc+}\end{array} \tag{A.3.4}$$

Inserting $\Omega_{abc\pm}$ from Eqn.(A.3.1) into Eqn.(A.3.1), we calculate $\Omega_{ab\mu\pm}$ and then $(D\Omega_{ab})_{cd+}\eta^{ad}\eta^{bc}$, giving the following final expression

$$\mathcal{L}_{R1} = (16\pi G_N)^{-1}(D\Omega_{ab})_{cd+}\eta^{ad}\eta^{bc}$$

$$\begin{array}{rcl}\mathcal{L}_{R1} & = & (64\pi G_N)^{-1}(e_{d\tau} + s_{d\lambda}\upsilon^\lambda_\tau)[(2X^{ab[\rho,\tau]}\eta^{cd} - X^{bc[\rho,\tau]}\eta^{ad})\partial_\rho(X^{[\mu,\nu]}_{bc}P_{a\mu\nu} - Y^{[\mu,\nu]}Q_{a\mu\nu}) \\ & & -(2Y^{ab[\rho,\tau]}\eta^{cd} - Y^{bc[\rho,\tau]}\eta^{ad})\partial_\rho(X^{[\mu,\nu]}_{bc}Q_{a\mu\nu} + Y^{[\mu,\nu]}_{bc}P_{a\mu\nu})] \\ & & +(64\pi G_N)^{-1}(X^{bc[\rho,\tau]}X^{[\mu,\nu]}_{bc}\eta^{ad} + 2X^{ab[\rho,\tau]}X^{d\ [\mu,\nu]}_{\ b})(-P_{a\mu\nu}\partial_\rho(e_{d\tau} + s_{d\lambda}\upsilon^\lambda_\tau) \\ & & +Q_{a\mu\nu}\partial_\rho s_{d\tau}) + (64\pi G_N)^{-1}(X^{bc[\rho,\tau]}Y^{[\mu,\nu]}_{bc}\eta^{ad} + 2X^{ab[\rho,\tau]}Y^{d\ [\mu,\nu]}_{\ b} \\ & & +Y^{bc[\rho,\tau]}X^{[\mu,\nu]}_{bc}\eta^{ad} + 2Y^{ab[\rho,\tau]}X^{d\ [\mu,\nu]}_{\ b})(P_{a\mu\nu}\partial_\rho s_{d\tau} + Q_{a\mu\nu}\partial_\rho(e_{d\tau} + s_{d\lambda}\upsilon^\lambda_\tau)) \\ & & (64\pi G_N)^{-1}(Y^{bc[\rho,\tau]}Y^{[\mu,\nu]}_{bc}\eta^{ad} + 2Y^{ab[\rho,\tau]}Y^{d\ [\mu,\nu]}_{\ b}) \\ & & (P_{a\mu\nu}\partial_\rho(e_{d\tau} + s_{d\lambda}\upsilon^\lambda_\tau) + Q_{a\mu\nu}\partial_\rho s_{d\tau}) \\ & & +m(32\pi G_N)^{-1}a_{\tau-}[((X^{bc[\rho,\tau]}X^{[\mu,\nu]}_{bc} - Y^{bc[\rho,\tau]}Y^{[\mu,\nu]}_{bc})\eta^{ad} - 2(X^{ab[\rho,\tau]}X^{d\ [\mu,\nu]}_{\ b} \\ & & -Y^{ab[\rho,\tau]}X^{d\ [\mu,\nu]}_{-b}))(P_{a\mu\nu}s_{d\rho} + Q_{a\mu\nu}(e_{d\rho} + s_{d\lambda}\upsilon^\lambda_\rho)) \\ & & +((Y^{bc[\rho,\tau]}X^{[\mu,\nu]}_{bc} - X^{bc[\rho,\tau]}Y^{[\mu,\nu]}_{bc})\eta^{ad} + 2(Y^{ab[\rho,\tau]}X^{d\ [\mu,\nu]}_{\ b} \\ & & +X^{ab[\rho,\tau]}X^{d\ [\mu,\nu]}_{\ b}))(P_{a\mu\nu}(e_{d\rho} + s_{d\lambda}\upsilon^\lambda_\rho) + Q_{a\mu\nu}s_{d\rho})].\end{array} \tag{A.3.5}$$

$$\begin{array}{rcl}\mathcal{L}_{R2} & = & (16\pi G_N)^{-1}((D\Gamma_a)_{ec+}(D\Gamma_b)_{fd+} + (D\Gamma_a)_{ec-}(D\Gamma_b)_{fd-}) \\ & & (\eta^{ab}\eta^{cd}\eta^{ef} + 3\eta^{ad}\eta^{be}\eta^{cf} + \eta^{ac}\eta^{bd}\eta^{ef}) \\ & = & (64\pi G_N)^{-1}(\eta^{ab}X^{cd[\mu,\nu]}X^{[\rho,\tau]}_{cd} - 3X^{bc[\mu,\nu]}X^{a\ [\rho,\tau]}_{\ c} + X^{ac[\mu,\nu]}X^{b\ [\rho,\tau]}_{\ c}) \\ & & [(P_{a\mu\nu}P_{b\rho\tau} - Q_{a\mu\nu}Q_{b\rho\tau})] \\ & & +(64\pi G_N)^{-1}(\eta^{ab}Y^{cd[\mu,\nu]}Y^{[\rho,\tau]}_{cd} - 3Y^{bc[\mu,\nu]}Y^{a\ [\rho,\tau]}_{\ b} + Y^{ac[\mu,\nu]}Y^{b\ [\rho,\tau]}_{\ c}) \\ & & [-P_{a\mu\nu}P_{b\rho\tau} + Q_{a\mu\nu}Q_{b\rho\tau}] \\ & & +(64\pi G_N)^{-1}(-\eta^{ab}(X^{cd[\mu,\nu]}Y^{[\rho,\tau]}_{cd} - X^{cd[\rho,\tau]}Y^{[\mu,\nu]}_{cd}) + 2(X^{bc[\mu,\nu]}Y^{a\ [\rho,\tau]}_{\ c} \\ & & -X^{bc[\rho,\tau]}Y^{a\ [\mu,\nu]}_{\ c}))[P_{a\mu\nu}Q_{b\rho\tau} - Q_{a\mu\nu}P_{b\rho\tau}],\end{array} \tag{A.3.6}$$

$$
\begin{aligned}
\mathcal{L}_{R3} &= (8\pi G_N)^{-1}[(D\Omega_{a\dot{5}})_{\dot{5}d+}\eta^{ad} + 4((D\Gamma_a)_{\dot{5}d+}(D\Gamma_b)_{\dot{5}c+} \\
&\quad +(D\Gamma_a)_{\dot{5}d-}(D\Gamma_b)_{\dot{5}c-})(\eta^{ad}\eta^{bc} - \eta^{ac}\eta^{bd})] \\
&= -m^2(4\pi G_N)^{-1}(\phi_+^2 + \phi_-^2)^{-2}[(\phi_+^2 - \phi_-^2)(s^{\mu\nu}s_{\mu\nu} - s^{\mu}_{\mu}s^{\nu}_{\nu}) \\
&\quad +\phi_+(\phi_+(v^{\mu\nu}s_{\mu\rho}v^{\rho\tau}s_{\tau\nu} - v^{\mu\nu}s_{\mu\nu}) + \phi_- s^{\mu\nu}s^{\rho}_{\mu}s_{\nu\rho})].
\end{aligned}
\qquad (A.3.7)
$$

A.4 Torsion Lagrangian

The contributions of the torsion to the total Lagrangian arise from the components $T_{\dot{5}bc}$, $T_{\dot{5}\dot{5}b}$ and $T_{\dot{5}\dot{5}\dot{5}}$.

With the physical reasoning that the gravity tensor $v_{\mu\nu}$ represents a field of spin 2, we assume that the gravity tensor field $v_{\mu\nu}$ is symmetric in the indices μ, ν, i.e. $v_{\mu\nu} = v_{\nu\mu}$. This also implies $s_{\mu\nu} = s_{\nu\mu}$ and hence,

$$
(D\Gamma_b)_{\dot{5}c\pm} - (D\Gamma_c)_{\dot{5}b\pm} = 0, \qquad (A.4.1)
$$

Hence, the projections of the torsion components can be expressed in terms of $D\Gamma$ as follows

$$
\begin{aligned}
(T_{\dot{5}bc})_{\pm} &= (D\Gamma_{\dot{5}})_{bc\pm}, \\
(T_{\dot{5}b\dot{5}})_{\pm} &= (D\Gamma_{\dot{5}})_{\dot{5}b\pm}, \\
(T_{\dot{5}\dot{5}\dot{5}})_{\pm} &= (D\Gamma_{\dot{5}})_{\dot{5}\dot{5}\pm},
\end{aligned}
\qquad (A.4.2)
$$

Using the expressions of $D\Gamma$ in Appendix A.2, we obtain

$$
\mathcal{L}_{T1} = (4g_V^2)^{-1}[(T_{\dot{5}bc})_+(T^{\dot{5}bc})_+ - (T_{\dot{5}bc})_-(T^{\dot{5}bc})_-] \ , \qquad (A.4.3)
$$

$$
\begin{aligned}
\mathcal{L}_{T1} &= -(4g_V^2)^{-1}(X^{[\mu,\nu]}_{bc}X^{bc[\rho,\tau]} + Y^{[\mu,\nu]}_{bc}Y^{bc[\rho,\tau]})[(\phi_+^2 + \phi_-^2)(\tfrac{1}{4}a_{\mu\nu+}a_{\rho\tau_+} + \tfrac{1}{4}a_{\mu\nu-}a_{\rho\tau-} \\
&\quad +m^2 a_{\mu-}a_{\nu+}a_{\rho-}a_{\tau+}) + m a_{\rho-}a_{\tau+}(2a_{\mu\nu+}\phi_+\phi_- + a_{\mu\nu-}(\phi_+^2 - \phi_-^2))] \\
&\quad +m(g_V^2)^{-1}X^{[\mu,\nu]}_{bc}Y^{bc[\rho,\tau]}[(\phi_+^2 - \phi_-^2)(a_{\mu\nu+}a_{\tau-}a_{\rho+} + a_{\nu+}a_{\mu-}a_{\rho\tau+}) \\
&\quad +2\phi_+\phi_-(a_{\mu\nu-}a_{\tau+}a_{\rho-} + a_{\rho\tau-}a_{\nu-}a_{\mu+})],
\end{aligned}
\qquad (A.4.4)
$$

$$
\mathcal{L}_{T2} = (2g_V^2)^{-1}[(T_{\dot{5}\dot{5}c})_+(T^{\dot{5}\dot{5}c})_+ - (T_{\dot{5}\dot{5}c})_-(T^{\dot{5}\dot{5}c})_-], \qquad (A.4.5)
$$

$$
\begin{aligned}
\mathcal{L}_{T2} &= -(2g_V^2)^{-1}(\phi_+^2 + \phi_-^2)^{-1}[\tfrac{1}{4}g^{\mu\nu}(\partial_\mu\phi_+\partial_\nu\phi_+ + \partial_\mu\phi_-\partial_\nu\phi_-) + m^2(a^{\mu}_{-}a_{\mu-}(4 - 3\phi_+^2) \\
&\quad +\phi_-^2 a^{\mu}_{+}a_{\mu+} - 2\phi_+\phi_- a_{\mu-}a^{\mu}_{+} + 4a_{\mu+}v^{\mu}_{\rho}(v^{\rho\lambda}a_{\lambda+} - a^{\rho}_{+}\phi_+\phi_- + 2a^{\rho}_{-} - a^{\rho}_{-}\phi_+^2)) \\
&\quad +m(a^{\mu}_{-}\phi_+ + 2a_{\rho+}v^{\rho\mu}\phi_+ - a^{\mu}_{+}\phi_-)\partial_\mu\phi_-]
\end{aligned}
\qquad (A.4.6)
$$

$$\mathcal{L}_{T3} \;=\; -(4g_V^2)^{-1}(T_{\dot{5}\dot{5}\dot{5}})_+ T_+^{\dot{5}\dot{5}\dot{5}} \;=\; -\frac{m^2}{g_V^2 k^2}\phi_+^2(\phi_+^2+\phi_-^2)^{-2}. \qquad (A.4.7)$$

A.5 Gauge Lagrangian

The Lagrangian terms in Eqn.(6.8) can be calculated as follows

$$\mathcal{L}_{G1} = -\frac{1}{4g^2}(g^{\mu\rho}+g^{\sigma\lambda}v_\sigma^\mu v_\lambda^\rho)(g^{\nu\tau}+g^{\sigma\lambda}v_\sigma^\nu v_\lambda^\tau)(b_{\mu\nu+}b_{\rho\tau+}+b_{\mu\nu-}b_{\rho\tau-}) \quad (A.5.1)$$

$$\begin{aligned}\mathcal{L}_{G2} \;=\; & -\tfrac{1}{g^2}(\phi_+^2+\phi_-^2)^{-1}(g^{\mu\nu}\mathcal{D}_\mu\eta\mathcal{D}_\nu\bar{\eta} \\ & +2ig^\mu\sigma v_\sigma^\nu b_{\mu+}((\bar{\eta}-m)\mathcal{D}_\mu\eta-(\eta-m)\mathcal{D}_\mu\bar{\eta}) \\ & +4g^{\sigma\tau}v_\sigma^\mu v_\tau^\nu b_{\mu+}b_{\nu+}(\eta-m)(\bar{\eta}-m))\end{aligned} \qquad (A.5.2)$$

$$\mathcal{L}_{G3} = \frac{1}{2g^2}(\phi_+^2+\phi_-)^{-1}(\bar{\eta}\eta-m^2)^2. \qquad (A.5.3)$$

$$\begin{aligned}\mathcal{L}_{G4} \;=\; & \tfrac{1}{g^2}(X_{ab}^{[\mu,\nu]}X^{ab[\rho,\tau]}+Y_{ab}^{[\mu,\nu]}Y^{ab[\rho,\tau]}) \\ & [((b_{\mu\nu+}-ib_{\mu\nu-})a_\rho(\mathcal{D}_\tau\eta+2i(\eta-m)b_{\tau-})+c.c) \\ & +(\tfrac{1}{2}a_\mu a_\rho^*(\mathcal{D}_\nu\bar{\eta}(\mathcal{D}_\tau\eta+2i(\eta-m)b_{\tau-})+4b_{\nu-}(\eta-m)(\bar{\eta}-m))+c.c)] \\ & -\tfrac{1}{2g^2}(\phi_+^2+\phi_-^2)^{-1}g^{\mu\nu}[(X_\mu+X_\mu^*)(Y_\nu+Y_\nu^*)+(X_\mu-X_\mu^*+4imv_\mu^\rho b_{\rho+})(Z_\nu-Z_\nu^*) \\ & +\tfrac{1}{2}(Y_\mu+Y_\mu^*)(Y_\nu+Y_\nu^*)-\tfrac{1}{2}(Z_\mu-Z_\mu^*)(Z_\nu-Z_\nu^*)]\end{aligned} \qquad (A.5.4)$$

where

$$\begin{aligned} a_\mu &= a_{\mu+}+ia_{\mu-} \\ X_\mu &= (\mathcal{D}_\mu-2iv_\mu^\nu b_{\nu+})\eta \\ Y_\mu &= (a_\mu+iv_\mu^\nu a_\nu)(\eta-\bar{\eta})(\eta-m) \\ Z_\mu &= (a_\mu-iv_\mu^\nu a_\nu)(\eta+m)(\eta+\bar{\eta}-2m),\end{aligned} \qquad (A.5.5)$$

$c.c$ denotes the complex conjugates.

A.6 Fermionic Lagrangian

The following identities will be useful for the calculations of the Lagrangian in Eqn.(6.9).

$$\begin{aligned}\bar{\Psi}\Gamma^a A\Psi &= \bar{\psi}\gamma^a(A_+ + \gamma^5 A_-)\psi, \\ \bar{\Psi}\Gamma^{\dot{5}} A\Psi &= \bar{\psi}(\gamma^5 A_+ + A_-)\psi,\end{aligned} \qquad (A.6.1)$$

where

$$A = A_+ \mathbf{1} + A_- \mathbf{r}.$$

To prove these identities we can take the trace over the Z_2 indices then express ψ_L, ψ_R in terms of the 4-component Dirac spinor and the parity projection operator $\frac{1}{2}(1 \pm \gamma^5)$.

To calculate the terms $\mathcal{L}_{F1}$ and $\mathcal{L}_{F2}$ we use the following identity to decompose the product of three Γ matrices

$$\Gamma^a \Gamma^b \Gamma^c = \eta^{ab} \Gamma^c - \eta^{ac} \Gamma^b + \eta^{bc} \Gamma^a + i \epsilon^{abcd} \Gamma^{\dot{5}} \Gamma_d \tag{A.6.2}$$

from where we obtain the following simplification using the symmetric properties of the tensors Ω_{BCA}

$$\Gamma^a \Gamma^b \Gamma^c \Omega_{bca} \quad = \quad 2\eta^{ab} \Gamma^c \Omega_{bca} + i \epsilon^{abcd} \Gamma^{\dot{5}} \Gamma_d \Omega_{bca}. \tag{A.6.3}$$

The second term in Eqn (A.6.3) yields a vanishing contribution when inserted between spinors (Eqn (A.6.1)). Hence we can omit it and obtain,

$$\begin{array}{rcl} \Gamma^a A_1 & = & \Gamma^a E_a^\mu (D_\mu + i B_\mu) + \frac{i}{4} \Gamma^a \Gamma^b \Gamma^c \Omega^{(0)}_{bca} \\ & = & \Gamma^a [E_a^\mu (D_\mu + i B_\mu) - \frac{i}{2} \eta^{bc} \Omega^{(0)}_{abc}] \end{array} \tag{A.6.4}$$

$$\begin{array}{rcl} (A_1)_+ & = & e_a^\mu (\nabla_\mu + i b_{\mu+}) - i v_a^\mu b_{\mu-} - \frac{i}{2} (v_a^\mu v_b^\nu - v_a^\nu v_b^\mu) \partial_\mu e_\nu^b \\ & & + \frac{i}{2} (X_{ab}^{[\mu,\nu]} \partial_\mu (v_\nu^\lambda s_\lambda^b) + Y_{ab}^{[\mu,\nu]} \partial_\mu s_\nu^b), \\ (A_1)_- & = & v_a^\mu (i \partial_\mu - b_{\mu+}) - e_a^\mu b_{\mu-} \\ & & - \frac{1}{2} (X_{ab}^{[\mu,\nu]} \partial_\mu s_\nu^b + Y_{ab}^{[\mu,\nu]} \partial_\mu (e_\nu^b + v_\nu^\lambda s_\lambda^b)), \end{array} \tag{A.6.5}$$

where the general covariant derivative $\nabla_\mu = \partial_\mu + \frac{i}{2} e_b^\nu (\partial_\mu e_\mu^b - \partial_\nu e_\mu^b)$.

$$\Gamma^a A_2 = \frac{i}{4} \Gamma^a \Gamma^b \Gamma^c \Omega^{(1)}_{bca} = \frac{i}{2} \eta^{ab} \Gamma^c \Omega^{(1)}_{bca} \tag{A.6.6}$$

$$\begin{array}{rcl} (A_2)_+ & = & i m s_\mu^b [X_{ab}^{[\mu,\nu]} a_{\nu-} + Y_{ab}^{[\mu,\nu]} a_{\nu+}], \\ (A_2)_- & = & m s_\mu^b [X_{ab}^{[\mu,\nu]} a_{\nu+} - Y_{ab}^{[\mu,\nu]} a_{\nu-}], \end{array} \tag{A.6.7}$$

$$\Gamma^a A_3 = -\Gamma^a E_a^\mu A_\mu (D_5 + i B_5), \tag{A.6.8}$$

$$\begin{array}{rcl}(A_3)_+ &=& -\frac{i}{2}[(e^\mu_a - iv^\mu_a)\eta(a_{\mu+} + ia_{\mu-}) + (e^\mu_a + iv^\mu_a)\bar{\eta}(a_{\mu+} - ia_{\mu-})] \\ (A_3)_- &=& \frac{1}{2}(v^\mu_a - ie^\mu_a)\eta(a_{\mu+} + ia_{\mu-}) + \frac{1}{2}(v^\mu_a + ie^\mu_a)\bar{\eta}(a_{\mu+} - ia_{\mu-})\end{array} \tag{A.6.9}$$

$$\Gamma^{\dot{5}} A_4 = \frac{1}{2}\Gamma^{\dot{5}}[i(\Gamma^a\Gamma^b\Omega_{b\dot{5}a}) + 2\Phi^{-1}(D_5 + iB_5)], \tag{A.6.10}$$

Since the tensor $s^{\mu\nu}$ is symmetric, the terms, which contain $\gamma^a\gamma^b$, reduce to the trace of that tensor giving

$$\begin{array}{rcl}(A_4)_+ &=& -i(\phi_+^2 + \phi_-^2)^{-1}(\frac{1}{2}\eta(\phi_+ + i\phi_-) + \frac{1}{2}\bar{\eta}(\phi_+ - i\phi_-) - ms^\mu_\mu\phi_-) \\ (A_4)_- &=& (\phi_+^2 + \phi_-^2)^{-1}(\frac{i}{2}\eta(\phi_+ + i\phi_-) + -\frac{i}{2}\bar{\eta}(\phi_+ - i\phi_-) - ms^\mu_\mu\phi_+)\end{array} \tag{A.6.11}$$

Here, we give only the final results:

$$\begin{array}{rcl}\mathcal{L}_{F1} &=& i\bar{\Psi}(\Gamma^a E^\mu_a(D_\mu + iB_\mu) + \frac{i}{4}\Gamma^a\Gamma^b\Gamma^c\Omega^{(0)}_{bca}))\Psi \\ &=& \frac{i}{2}\bar{\psi}\gamma^a[e^\mu_a(\nabla_\mu + ib_{\mu+}) - iv^\mu_a b_{\mu-} + \frac{i}{2}(v^\mu_a v^\nu_b - v^\nu_a v^\mu_b)\partial_\mu e^b_\nu \\ && -\frac{i}{2}(X^{[\mu,\nu]}_{ab}\partial_\mu(v^\lambda_\nu s^b_\lambda) + Y^{[\mu,\nu]}_{ab}\partial_\mu s^b_\nu)]\psi \\ && +\frac{i}{2}\bar{\psi}\gamma^a\gamma^5[v^\mu_a(i\partial_\mu - b_{\mu+}) - e^\mu_a b_{\mu-} \\ && +\frac{1}{2}(X^{[\mu,\nu]}_{ab}\partial_\mu s^b_\nu + Y^{[\mu,\nu]}_{ab}\partial_\mu(e^b_\nu + v^\lambda_\nu s^b_\lambda))]\psi + h.c.\end{array} \tag{A.6.12}$$

$$\begin{array}{rcl}\mathcal{L}_{F2} &=& \frac{i}{4}\bar{\Psi}\Gamma^a\Gamma^b\Gamma^c\Omega^{(1)}_{bca}\Psi \\ &=& \frac{m}{2}\bar{\psi}\gamma^a(X^{[\mu,\nu]}_{ab} + i\gamma^5 Y^{[\mu,\nu]}_{ab})(i\gamma^5 a_{\nu+} - a_{\nu-})s^b_\mu\psi + h.c.\end{array} \tag{A.6.13}$$

$$\begin{array}{rcl}\mathcal{L}_{F3} &=& i\bar{\Psi}\Gamma^a E^\mu_a A_\mu(D_5 + iB_5)\Psi \\ &=& -\frac{1}{2}\bar{\psi}\gamma^a\eta(a_{\mu+} + ia_{\mu-})[(e^\mu_a - iv^\mu_a) + \gamma^5(e^\mu_a + iv^\mu_a)]\psi + h.c.\end{array} \tag{A.6.14}$$

$$\begin{array}{rcl}\mathcal{L}_{F4} &=& \frac{i}{2}\bar{\Psi}\Gamma^{\dot{5}}[(\Gamma^a\Gamma^b\Omega_{b\dot{5}a}) + 2\Phi^{-1}(D_5 + iB_5)]\Psi \\ &=& -\frac{i}{2}\bar{\psi}(\phi_+^2 + \phi_-^2)^{-1}\eta(\phi_+ + i\phi_-)(1 - \gamma^5)\psi + h.c.\end{array} \tag{A.6.15}$$

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